DIFFERENTIAL EQUATIONS

Equations Reducible to Exact

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Outline

- Exact Differential Equations

Integrating Factor

For the differential equation $M(x, y)dx + N(x, y)dy = 0$, a nonzero function $\mu(x, y)$ is called an integrating factor if the differential equation

$$\mu M(x, y)dx + \mu N(x, y)dy = 0$$

is exact.

Example

he equation $xdy + 2ydx = 0$ becomes exact on multiplication with x. That is the equation $2xydx + x^2dy = 0$ is exact. Because, $d(x^2y) = 2xydx + x^2dy = 0$.

Examples

- The equation $ydx - xdy = 0$ becomes exact on multiplication with x. That is the equation $2xydx + x^2dy = 0$ is exact. Because, $d(x^2y) = 2xydx + x^2dy = 0$.
- The equation $ydx - xdy = 0$ becomes exact on multiplication with x. That is the equation $2xydx + x^2dy = 0$ is exact. Because, $d(x^2y) = 2xydx + x^2dy = 0$.

How to find an integrating factor?

If $\mu(x, y)$ is an integrating factor if the differential equation

$$\mu M(x, y)dx + \mu N(x, y)dy = 0$$

is exact. By an earlier theorem we must have

use product rule

$$\frac{\partial}{\partial y}(\mu M(x, y)) = \frac{\partial}{\partial x}(\mu N(x, y))$$

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y}M - \frac{\partial \mu}{\partial x}N = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

A solution of above gives an integrating factor. But solving this PDE is not an easy task.

Solution in special cases!

However, we can find an integrating factor of $Mdx + Ndy = 0$ by solving the equation

$$\frac{\partial \mu}{\partial y}M - \frac{\partial \mu}{\partial x}N = \mu \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

in some special cases

- M and N are homogenous functions of same degree.
- M and N are in the form of $M = f(xy)y$ and $N = g(xy)x$.
- $\mu(x, y) = \mu(x)$ and $\mu(x, y) = \mu(y)$.

Integrating factor when M and N are homogeneous function of same degree

Problem 1: Solve $(x^2y)\,dx - (x^3 + y^3)\,dy = 0.$

Solution

Here, $M = x^2y$ and $N = -x^3 - y^3$ are homogeneous functions of degree 3. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(x^2y\right) = x^2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-x^3 - y^3\right) = -3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{differential equation is non-exact.}$$

$$\frac{1}{Mx + Ny} = \frac{1}{x(x^2y) - y(x^3 + y^3)}$$

$$= \frac{1}{x^3y - x^3y - y^4} = \frac{-1}{y^4}$$

Cont...

multiply the given differential equation with $\frac{1}{y^4}$ we get

$$\frac{1}{y^4}\, x^2 y \; dx - \frac{1}{y^4}(x^3 + y^3)dy = 0$$

$$\frac{x^2}{y^3} \; dx - \left(\frac{x^3}{y^4} + \frac{1}{y} \right) dy = 0$$

Which is exact and hence its solution is

$$\int \frac{x^2}{y^3} \; dx - \int \frac{1}{y} dy = c$$

$$\frac{1}{y^3} \int x^2 \, dx - \log y = c$$

$$\frac{x^3}{3y^3} - \log y = c$$

Problem 2: Solve $(y^2)\,dx + (x^2 - xy - y^2)\,dy = 0.$

Solution

Here, $M = y^2$ and $N = x^2 - xy - y^2$ are homogeneous functions of degree 2. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y^2\right) = 2y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x^2 - xy - y^2\right) = 2x - y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{differential equation is non-exact.}$$

$$\frac{1}{Mx + Ny} = \frac{1}{x(y^2) + y(x^2 - xy - y^2)}$$

$$= \frac{1}{xy^2 + yx^2 - xy^2 - y^3} = \frac{1}{yx^2 - y^3} = \frac{1}{y(x^2 - y^2)}$$

Cont...

multiply the given differential equation with $\frac{1}{y(x^2-y^2)}$ we get

$$\frac{1}{y(x^2-y^2)}\, y^2\, dx + \frac{1}{y(x^2-y^2)}(x^2-xy-y^2)dy = 0$$

$$\frac{y}{x^2-y^2}\, dx - \left(\frac{1}{y} + \frac{x}{x^2-y^2}\right) dy = 0$$

Which is exact and hence its solution is

$$\int \frac{y}{x^2-y^2}\, dx + \int \frac{1}{y}\, dy = c$$

$$y \int \frac{1}{x^2-y^2}\, dx + \log y = c$$

$$y \times \frac{1}{2y} \log\left(\frac{x-y}{x+y}\right) + \log y = c$$

$$\frac{1}{2} \log\left(\frac{x-y}{x+y}\right) + \log y = c$$

Theorem

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ *is continuous and function of* x *then*

$$\mu(x) = \exp\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx$$

is an integrating factor of $M(x,y)dx + N(x,y)dy = 0$.

Proof.

If $\mu(x,y)$ is an integrating factor, we must have

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \Rightarrow \frac{\partial \mu}{\partial y}M - \frac{\partial \mu}{\partial x}N = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

If $\mu = \mu(x)$ then $\frac{\partial \mu}{\partial y} = 0$ and $-\frac{\partial \mu}{\partial x}N = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$. $\frac{\partial \mu}{\mu} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx$. On integration we get desired result.

Problem 1: Solve $y + \frac{y^3}{3} + \frac{x^2}{2} \quad dx + \frac{1}{4}(x + xy^2)\,dy = 0$

Solution

Here, $M = y + \dfrac{y^3}{3} + \dfrac{x^2}{2}$ and $N = \dfrac{1}{4} \; x + xy^2$. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \quad y + \frac{y^3}{3} + \frac{x^2}{2} \quad = 1 + y^2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \; \frac{1}{4} \; x + xy^2 \quad = \frac{1}{4} \; 1 + y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{differential equation is non-exact.}$$

$$\frac{1}{N} \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{4}{x + xy^2} \quad 1 + y^2 - \frac{1}{4}(1 + y^2) \quad = \frac{3}{x}$$

$$= \frac{4}{x(1 + y^2)} \quad \frac{3(1 + y^2)}{4} \quad = \frac{3}{x}$$

Count….

- I.F= $\exp\left(\int \frac{3}{x}dx\right) = \exp(3\log x) = \exp(\log x^3) = x^3.$

multiply the given differential equation with x^3 we get

$$\left(x^3y + \frac{x^3y^3}{3} + \frac{x^5}{2}\right)dx + \frac{1}{4}\left(x^4 + x^4y^2\right)dy = 0$$

Which is exact and hence its solution is

$$\int\left(x^3y + \frac{x^3y^3}{3} + \frac{x^5}{2}\right)dx + \int 0dy = c$$

$$y\int x^3dx + \frac{y^3}{3}\int x^3dx + \frac{1}{2}\int x^5dx = c$$

$$\frac{x^4y}{4} + \frac{x^4y^3}{12} + \frac{x^6}{12} = \frac{c}{12} \Rightarrow 3x^4y + x^4y^3 + x^6 = c$$

Problem 2: Solve $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0$.

Solution

Here, $M = x^2 + y^2 + 1$ and $N = -2xy$. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 + 1\right) = 2y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-2xy) = -2y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{differential equation is non-exact.}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-1}{2xy}(2y - (-2y))$$

$$= \frac{-4y}{2xy} = \frac{-2}{x}$$

Cont...

I.F= $\exp \int \frac{-2}{x} dx = \exp(-2\log x) = \exp(\log x^{-2}) = \frac{1}{x^2}.$

multiply the given differential equation with $\frac{1}{x^2}$. we get

$$\frac{x^2 + y^2 + 1}{x^2} dx + \frac{-2xy}{x^2} dy = 0$$

Which is exact and hence its solution is

$$\int \frac{x^2 + y^2 + 1}{x^2} dx + \int 0dy = c$$

$$\int 1dx + (y^2 + 1) \int \frac{1}{x^2} dx = c$$

$$x + (y^2 + 1) \frac{-1}{x} = c \Rightarrow x^2 - y^2 - 1 = cx$$

Problem 3: Solve $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$.

Solution

Here, $M = x^2 + y^2 + 2x$ and $N = 2y$. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 + 2x\right) = 2y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2y) = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{differential equation is non-exact.}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2y}(2y - 0) = 1$$

Cont…

$$\text{I.F} = \exp\left(\int 1\,dx\right) = \exp(x) = e^x.$$

multiply the given differential equation with e^x we get

$$e^x\left(x^2 + y^2 + 2x\right)dx + 2ye^x dy = 0$$

Which is exact and hence its solution is

$$\int \left(e^x x^2 + y^2 e^x + 2xe^x\right)dx + \int 0\,dy = c$$

$$\int e^x(x^2 + 2x)dx + y^2 \int e^x dx = c$$

$$e^x x^2 + y^2 e^x = c \Rightarrow e^x(x^2 + y^2) = c$$

Theorem

If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is continuous and function of y then

$$\mu(y)=\exp\int\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dy$$

is an integrating factor of $M(x,y)dx+N(x,y)dy=0$.

Proof.

If $\mu(x,y)$ is an integrating factor, we must have

$$\frac{\partial}{\partial y}(\mu M)=\frac{\partial}{\partial x}(\mu N)\Rightarrow\frac{\partial \mu}{\partial y}M-\frac{\partial \mu}{\partial x}N=\mu\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$$

If $\mu=\mu(y)$ then $\frac{\partial \mu}{\partial x}=0$ and $\frac{\partial \mu}{\partial y}M=\mu\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$.

$\frac{\partial \mu}{\mu}=\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dy$. On integration we get desired result.

Problem 1: Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$

Solution

Here, $M = y^4 + 2y$ and $N = xy^3 + 2y^4 - 4x.$ Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y^4 + 2y\right) = 4y^3 + 2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(xy^3 + 2y^4 - 4x\right) = y^3 - 4$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{differential equation is non-exact.}$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y^4 + 2y}\left(y^3 - 4 - 4y^3 - 2\right)$$

$$= \frac{-3y^3 - 6}{y(y^3 + 2)} = \frac{-3(y^3 + 2)}{y(y^3 + 2)} = \frac{-3}{y}$$

Cont...

$$\text{I.F}= \exp\int \frac{-3}{y}dy = \exp(-3\log y) = \exp(\log y^{-3}) = \frac{1}{y^3}.$$

multiply the given differential equation with e^x we get

$$\frac{1}{y^3}\left(y^4 + 2y\right)dx + \frac{1}{y^3}(xy^3 + 2y^4 - 4x)dy = 0$$

Which is exact and hence its solution is

$$\int \frac{1}{y^3}\left(y^4 + 2y\right)dx + \int \frac{2y^4}{y^3}dy = c$$

$$\frac{1}{y^3}\left(y^4 + 2y\right)\int 1\,dx + 2\int y\,dy = c$$

$$\frac{1}{y^3}\left(y^4 + 2y\right)x + \frac{2y^2}{2} = c \Rightarrow xy + \frac{2x}{y^2} + y^2 = c$$

Problem 2: Solve $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0.$

Solution

Here, $M = xy^3 + y$ and $N = 2\left(x^2y^2 + x + y^4\right)$. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(xy^3 + y\right) = 3xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2\frac{\partial}{\partial x}\left(x^2y^2 + x + y^4\right) = 2(2xy^2 + 1)$$

$$= 4xy^2 + 2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{differential equation is non-exact.}$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^3 + y}\left(4xy^2 + 2 - 3xy^2 - 1\right)$$

$$= \frac{xy^2 + 1}{y(xy^2 + 1)} = \frac{1}{y}$$

Cont...

$$\text{I.F} = \exp\left(\int \frac{1}{y}\,dy\right) = \exp(\log y) = y.$$

multiply the given differential equation with y we get

$$y\left(xy^3 + y\right)dx + 2y(x^2y^2 + x + y^4)dy = 0$$
$$\left(xy^4 + y^2\right)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0$$

Which is exact and hence its solution is

$$\int\left(xy^4 + y^2\right)dx + \int 2y^5 dy = c$$
$$y^4\int x\,dx + y^2\int 1\,dx + \frac{2y^6}{6} = 3$$
$$\frac{y^4x^2}{2} + y^2x + \frac{y^6}{3} = c$$

through Exact

Consider the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{rewrite the equation as}$$

$$(P(x)y - Q(x))\,dx + dy = 0$$

Here, $M = P(x)y - Q(x)$ and $N = 1$. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(P(x)y - Q(x)) = P(x)$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1) = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{differential equation is non-exact.}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{1}(P(x) - 0) = P(x)$$

$$\text{I.F} = \exp\left(\int P(x)\,dx\right).$$

multiply the given differential equation with $I.F = \exp\left(\int P(x)\,dx\right)$ we get

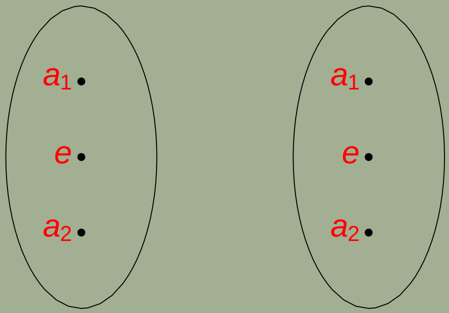
$$I.F\,(P(x)y - Q(x))\,dx + \text{I.F}\,dy = 0$$

Which is exact and hence its solution is

$$\int \text{I.F}\,(P(x)y - Q(x))\,dx + \int 0\,dy = c$$

$$\int \text{I.F}\,P(x)y\,dx - \int \text{I.F}\,Q(x)\,dx = c$$

$$y\,\text{I.F} = \int \text{I.F}\,Q(x)\,dx + c$$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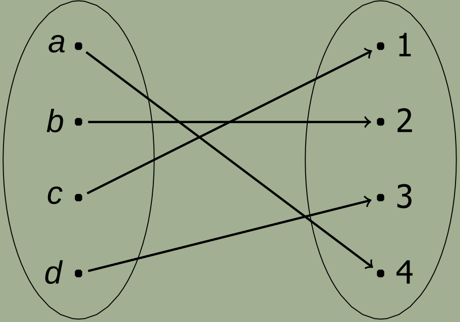


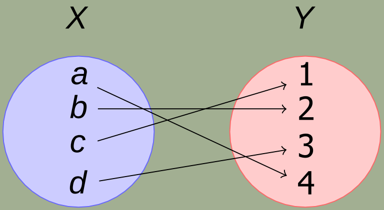
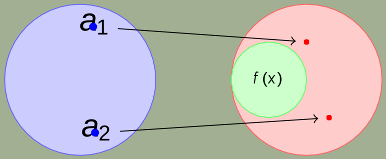


Figure: Mapping diagram of relation S



$$f : X \to Y$$

1. Solid Geometry

The Cone

A cone is a surface generated by a straight line which passes through a fixed point and intersects with a given curve.
here

- The fixed point is called the **vertex**.

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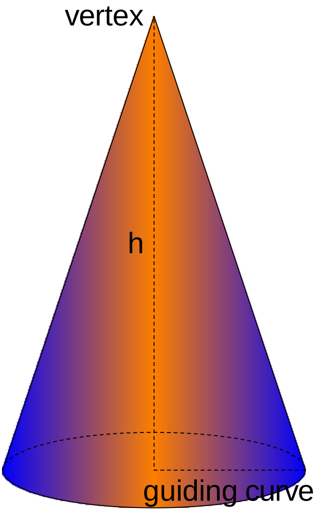
- The fixed point is called the **vertex**.
- The straight lines are called a **generators**.
- The given curve is called a **guiding curve**.

cone



Definition

A polynomial of three variables $f(x, y, z)$ is called a homogeneous polynomial of degree n if

$$f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$$

.

Examples:

1. $f(x, y, z) = x^3 + xy^2 + z^3$ is a homogeneous polynomial of degree 3

Homogeneous Equations

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Examples:

1. $f(x, y, z) = x^3 + xy^2 + z^3$ is a homogeneous polynomial of degree 3
2. $f(x, y, z) = xy + yz + zx$ is a homogeneous polynomial of degree 2.

Homogeneous Equations

Definition

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$$f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$$

.

Examples:

1. $f(x, y, z) = x^3 + xy^2 + z^3$ is a homogeneous polynomial of degree 3
2. $f(x, y, z) = xy + yz + zx$ is a homogeneous polynomial of degree 2.
3. $f(x, y, z) = x^3 + xyz + zx$ is not a homogeneous polynomial.

Theorem

Every Homogeneous Polynomial of degree n is a cone with ***vertex at origin.***

The equation $f(x, y, z) = xy + yz + zx = 0$ is a cone with vertex at origin.

Theorem

The line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

is generator of the cone $f(x, y, z) = 0$ *if* $f(l, m, n) = 0$

Cone Through Coordinate Axes

Theorem

The general equation of the cone through the coordinate axes is

$$fyz + gzx + hxy = 0$$

Proof:
The general second degree equation is

$$f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad (1)$$

since $X-$ axis is a generator, then $f(1, 0, 0) = 0$ which implies $a = 0$.

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since $X-$ axis is a generator, then $f(1, 0, 0) = 0$ which implies $a = 0$.
since $Y-$ axis is a generator, then $f(0, 1, 0) = 0$ which implies $b = 0$

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since $X-$ axis is a generator, then $f(1, 0, 0) = 0$ which implies $a = 0$.
since $Y-$ axis is a generator, then $f(0, 1, 0) = 0$ which implies $b = 0$
and $Z-$ axis is a generator, then $f(0, 0, 1) = 0$ which implies $c = 0$.
Hence, equation of the cone is

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since $X-$ axis is a generator, then $f(1, 0, 0) = 0$ which implies $a = 0$.
since $Y-$ axis is a generator, then $f(0, 1, 0) = 0$ which implies $b = 0$
and $Z-$ axis is a generator, then $f(0, 0, 1) = 0$ which implies $c = 0$.
Hence, equation of the cone is

$$\boxed{fyz + gzx + hxy = 0}$$

Example

Find the equation of the cone which passes through the coordinate axes and the lines

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3} \quad \text{and} \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{1}$$

Example

Find the equation of the cone which passes through the coordinate axes and the lines

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3} \quad \text{and} \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{1}$$

Equation of the cone passes through the coordinate axes is

$$fyz + gzx + hxy = 0 \tag{2}$$

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Equation of the cone passes through the coordinate axes is

$$fyz + gzx + hxy = 0 \tag{2}$$

the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ lies on (2),

$$f(-2)(3) + g(3)(1) + h(1)(-2) = 0 \Rightarrow 6f + 3g - 2h = 0 \tag{3}$$

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$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3} \quad \text{and} \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{1}$$

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$$f(-2)(3) + g(3)(1) + h(1)(-2) = 0 \Rightarrow 6f + 3g - 2h = 0 \tag{3}$$

the line $\frac{x}{2} = \frac{y}{1} = \frac{z}{1}$ lies on (2),

$$f(1)(1) + g(1)(2) + h(1)(2) = 0 \Rightarrow f + 2g + 2h = 0 \qquad (4)$$

solving the equation (3) and (4)

Conti...

solving the equation (3) and (4)

$$\frac{f}{-6-4} = \frac{g}{2-12} = \frac{h}{12+3}$$

$$\frac{f}{2} = \frac{g}{2} = \frac{h}{-3}$$

hence the equation of the cone is

$$\boxed{2yz + 2zx - 3xy = 0}$$

Equation of cone with base curve

Theorem

The equation of the cone with vertex at (α, β, γ) *and the guiding curve is* $f(x, y) = 0, z = 0$ *is*

$$(z-\gamma)^2 f\left(\alpha - \gamma\frac{x-\alpha}{z-\gamma}, \beta - \gamma\frac{y-\beta}{z-\gamma}\right) = 0.$$

Equation of cone with base curve

Theorem

The equation of the cone with vertex at (α, β, γ) and the guiding curve is $f(x, y) = 0, z = 0$ is

$$(z-\gamma)^2 f\left(\alpha - \gamma\frac{x-\alpha}{z-\gamma}, \beta - \gamma\frac{y-\beta}{z-\gamma}\right) = 0.$$

Proof: Let the equation of the line through the point (α, β, γ) with direction cosines (l, m, n) is

Equation of cone with base curve

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Proof: Let the equation of the line through the point (α, β, γ) with direction cosines (l, m, n) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \tag{5}$$

and the point on the line (5) is $(lr + \alpha, mr + \beta, nr + \gamma)$

Conti...

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$$f\left(\alpha-\gamma\frac{l}{n}, \beta-\gamma\frac{m}{n}\right)=0 \tag{7}$$

eliminating l, m, n from (5) and (7) we get

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$$\boxed{f\left(\alpha - \gamma\frac{x - \alpha}{z - \gamma}, \beta - \gamma\frac{y - \beta}{z - \gamma}\right) = 0}$$

Example

Find the equation of the cone whose vertex at $(1, 1, 0)$ and whose guiding curve is $y = 0, x^2 + z^2 = 4$.

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Solution:

Equation to the generator through the vertex $(1, 1, 0)$ is

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z}{n} = r \tag{8}$$

and general point on the above generator is $(lr + 1, mr + 1, nr)$. This point lies on the guiding curve $y = 0, x^2 + z^2 = 4$, then we have $mr + 1 = 0, (lr + 1)^2 + (nr)^2 = 4$. From the equation $mr + 1 = 0$ we have $r = \frac{-1}{m}$. Plug r in to $(lr + 1)^2 + (nr)^2 = 4$ we get

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$$\left(1-\frac{l}{m}\right)^2 + \frac{n^2}{m^2} = 4 \Rightarrow (m-l)^2 + n^2 = 4m^2 \tag{9}$$

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conti...

eliminating l, m, n from (8) and (9) we get

$$
\begin{aligned}
1 - \frac{x-1}{y-1}^{2} + \frac{z}{y-1}^{2} &= 4 \\
(y-1-(x-1))^2 + z^2 &= 4(y-1)^2 \\
x^2 + y^2 - 2xy + z^2 &= 4(y^2 - 2y + 1)
\end{aligned}
$$

after simplification we have, required equation of cone is

$$\boxed{x^2 - 3y^2 + z^2 - 2xy + 8y - 4 = 0.}$$

Right Circular Cone

Definition

A right circular cone is a surface generated by a line which passe through a fixed point and makes an constant angle with a fixed line through a fixed point is called a right circular cone.

Here,

1. Fixed line is called **axis** of the cone.

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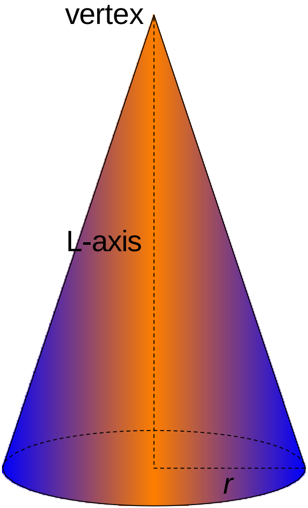
2. The constant angle θ is called the **semi-vertical angle.**

Right Circular Cone...



Equation of Right Circular Cone

Theorem

The equation of a right circular cone with vertex (α, β, γ), semi-vertical angle θ and axis having direction ratios (l, m, n) is

$$[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2$$
$$= \left(l^2 + m^2 + n^2\right)\left[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\right]\cos^2\theta$$

Proof:

Let $V = (\alpha, \beta, \gamma)$ be the vertex and let the direction ratios of the axis line L is (l, m, n). Let $P = (x, y, z)$ be any point on the cone. Then

$$\cos\theta = \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2} \times \sqrt{l^2 + m^2 + n^2}} \qquad (10)$$

Corollaries

1. If the vertex be the origin then the equation of right circular cone is

$$(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)\cos^2\theta.$$

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Example

Find the equation of the right circular cone whose vertex is the origin, axis as the line $x = t, y = 2t, z = 3t$ and whose semi-vertical angle is 60^0.

Solution:

given that vertex $V = (\alpha, \beta, \gamma) = (0, 0, 0)$ and semi-vertical angle $\theta = 60^0$, and equation of the axis

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = t$$

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$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = t$$

whose directional ratios are $(l, m, n) = (1, 2, 3)$. Required cone is

$$\begin{aligned} & \left[(x-0)^2 + (y-0)^2 + (z-0)^2\right](1^2 + 2^2 + 3^2)\cos^2 60^0 \\ = \; & [1(x-0) + 2(y-0) + 3(z-0)]^2 \end{aligned}$$

Conti...

$$\frac{14}{4}(x^2 + y^2 + z^2) = (x + 2y + 3z)^2$$

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$$7(x^2+y^2+z^2) = 2(x^2+4y^2+9z^2+4xy+12yz+6xy)$$

Therefore, equation of the cone is

Conti...

$$\frac{14}{4}(x^2 + y^2 + z^2) = (x + 2y + 3z)^2$$
$$7(x^2 + y^2 + z^2) = 2(x^2 + 4y^2 + 9z^2 + 4xy + 12yz + 6xy)$$

Therefore, equation of the cone is

$$\boxed{5x^2 - y^2 - 11z^2 - 24yz - 12zx - 8xy = 0}$$

Enveloping Cone

Definition

Let S be a surface and P be a point not on the surface. The set of tangent lines to the surface S and passes through P form a cone with vertex at P.

Enveloping Cone...

Theorem

The enveloping cone of the sphere $S = x^2 + y^2 + z^2 - a^2 = 0$ *with vertex at* (x_1, y_1, z_1) *is* $S_1^2 = SS_{11}$ *i.e.,*

$$(xx_1 + yy_1 + zz_1 - a^2)^2 = (x^2 + y^2 + z^2 - a^2)(x_1^2 + y_1^2 + z_1^2 - a^2)$$

Problem

Find the equation of enveloping cone of the sphere

$$x^2 + y^2 + z^2 + 2x - 2y = 2$$

with vertex at $(1, 1, 1)$.

Solution:

Given that, vertex $(1, 1, 1)$ and the equation of the sphere $S = x^2 + y^2 + z^2 + 2x - 2y - 2 = 0.$

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Find the equation of enveloping cone of the sphere

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with vertex at $(1, 1, 1)$.

Solution:

Given that, vertex $(1, 1, 1)$ and the equation of the sphere $S = x^2 + y^2 + z^2 + 2x - 2y - 2 = 0$. Now

$$S_1 = x1 + y1 + z1 + (x + 1) - (y + 1) - 2 \Rightarrow 2x + z - 2$$

and $S_{11} = 1 + 1 + 1 + 2 - 2 - 2 = 1$ equation of enveloping cone is $S_1^2 = SS_{11}$.

$$\begin{aligned}(2x + z - 2)^2 &= 1(x^2 + y^2 + z^2 + 2x - 2y - 2) \\ &= \boxed{3x^2 - y^2 + 4zx - 10x + 2y - 4z + 6 = 0}\end{aligned}$$

Thank You !