THE LAPLACE TRANSFORMATIONS

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Outline of the Talk

What we know is not much. What we do not know is immense.

Pierre-Simon Laplace

Definition and Examples

Definition

The Laplace transform is defined in the following way. Let $f(t)$ be defined for $t \geq 0$. Then the Laplace transform of f, which is denoted by $L\ [f(t)]$ or by $F(s)$, is defined by the following equation

$$L\ [f(t)] = F(s) = \lim_{n\to\infty} \int_o^n e^{-st} f(t)dt = \int_o^\infty e^{-st} f(t)dt$$

If the integral is converges.

Examples

Examples

Find the Laplace transform of the constant function $f(t) = 1, 0 \leq t < \infty$

$$F(s) = \int_o^\infty e^{-st} f(t)\,dt = \int_o^\infty e^{-st}(1)\,dt$$

$$= \lim_{b\to\infty} \int_o^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^b, \text{ provided } s \neq 0$$

$$= \lim_{b\to\infty} \frac{e^{-sb}}{-s} - \frac{1}{-s}$$

$$= \lim_{b\to\infty} \frac{e^{-sb}}{-s} + \frac{1}{s}$$

Examples

Recall a limit

$$e^{-x} \to \begin{cases} 0, & \text{if } x \to +\infty; \\ \infty, & \text{if } x \to -\infty. \end{cases}$$

Hence,

$$\lim_{b\to\infty} \frac{e^{-sb}}{-s} = \begin{cases} 0, & \text{if } s > 0; \\ \infty, & \text{if } s < 0. \end{cases} \quad (1)$$

Thus

$$F(s) = L[1] = \lim_{b\to\infty} \frac{e^{-sb}}{-s} + \frac{1}{s} = 0 + \frac{1}{s} = \frac{1}{s}, s > 0$$

$$\boxed{F(s) = L[1] = \frac{1}{s}, s > 0}$$

In this case domain of the transformation is the set of all positive real numbers.

Examples

Examples

Find the Laplace transform of the function $f(t) = e^{at}$, where a is constant, $0 \leq t < \infty$

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^{\infty} e^{-st}(e^{at})\,dt$$

$$= \lim_{b\to\infty} \int_0^{b} e^{-st+at}\,dt$$

$$= \lim_{b\to\infty} \int_0^{b} e^{-(s-a)t}\,dt, \quad Use \int e^{at}\,dt = \frac{e^{at}}{a}$$

$$= \lim_{b\to\infty} \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^b, \quad \text{provided } s \neq a$$

$$= \lim_{b\to\infty} \left[\frac{e^{-(s-a)b}}{-(s-a)} - \frac{1}{-(s-a)}\right]$$

by (1)

$$\lim_{b\to\infty}\left[\frac{e^{-(s-a)b}}{-(s-a)}+\frac{1}{s-a}\right]=0+\frac{1}{s-a}=\frac{1}{s-a},\ s>a.$$

$$L\,[e^{at}]=\frac{1}{s-a},\ s>a.$$

In this case domain of the transformation is the set of all positive real numbers greater than a. In particular

$$L\,[e^{t}] = \frac{1}{s-1},\ s>1$$

$$L\,[e^{-t}] = \frac{1}{s+1},\ s>-1$$

$$L\,[e^{2t}] = \frac{1}{s-2},\ s>2$$

Examples

Find the Laplace transform of the function $f(t) = \sin(at)$, where a is constant.

$$F(s) = \int_{o}^{\infty} e^{-st} (\sin at)\mathrm{dt}$$

$$= \lim_{b\to\infty} \frac{e^{-st}}{s^2+a^2}(-s\sin(at) - a\cos(at))\Big|_0^b$$

$$= \lim_{b\to\infty} \frac{e^{-sb}}{s^2+a^2}(-s\sin(ab) - a\cos(ab)) - \frac{-a}{s^2+a^2}$$

$$= \frac{a}{s^2+a^2}$$

$$\boxed{L\ \{\sin(at)\} = \frac{a}{s^2+a^2}, \quad s > 0}.$$

Examples

Find the Laplace transform of the function $f(t) = \cos(at)$, where a is constant.

$$F(s) = \int_o^\infty e^{-st} (\cos at)dt$$

$$= \lim_{b\to\infty} \left[\frac{e^{-st}}{s^2+a^2}(-s\cos(at) + a\sin(at))\right]_0^b$$

$$= \lim_{b\to\infty} \frac{e^{-sb}}{s^2+a^2}(-s\cos(ab) + a\sin(ab)) - \frac{-s}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$

$$\boxed{L\{\cos(at)\} = \frac{s}{s^2+a^2}, \quad s > 0}.$$

Find the Laplace transform of the function $f(t) = t^n$, where $n \in \mathbb{N}$.

Sol: We recall that

$$\int_0^\infty e^{-st}dt = \frac{1}{s}, \quad s > 0 \tag{2}$$

formally differentiate (2) with respect to s. This gives

$$\int_0^\infty te^{-st}dt = \frac{1}{s^2}, \quad s > 0 \tag{3}$$

which means that

$$L\,[t] = \frac{1}{s^2}, s > 0$$

formally differentiate (3) with respect to s. This gives

$$\int_0^\infty t^2e^{-st}dt = \frac{2}{s^3}, \quad s > 0 \tag{4}$$

which means that

$$L\,[t^2] = \frac{2}{s^3}, s > 0$$

formally differentiate (4) with respect to s. This gives

$$\int_0^\infty t^3 e^{-st} dt = \frac{6}{s^4}, \quad s > 0 \tag{5}$$

which means that

$$L\,[t^3] = \frac{3!}{s^4}, s > 0 \quad 3! = 6$$

Similarly, differentiation of (2) n times yields

$$\boxed{L\,[t^n] = \frac{n!}{s^{n+1}}, s > 0}$$

Laplace transform is Linear

If c is a constant and f and g are functions then

$$L\{cf(t)\} = cL\{f(t)\}$$
$$L\{cf(t)+g(t)\} = cL\{f(t)\}+L\{g(t)\}$$

More precisely,

Theorem

Suppose $F_1(s) = L\{f(t)\}$ exists for $s > a_1$ and $F_2(s) = L\{g(t)\}$ exists for $s > a_2$. Then

$$L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\} \quad (6)$$

for $s > a$, where $a = \max\{a_1, a_2\}$.

Important Note

Note that:

$$L\{f(t)g(t)\} \neq L\{f(t)\} \times L\{g(t)\}$$

Example

We know that

$$L\{t^2\} = \frac{2}{s^3}$$

but

$$L\{t\} \times L\{t\} = \frac{1}{s^2} \times \frac{1}{s^2} = \frac{1}{s^4} \neq \frac{2}{s^3}\,!!!$$

Examples on Linearity

$$L\{4t^5\} = 4L\{t^4\} = 4\frac{5!}{s^6} = \frac{480}{s^6}, s > 0$$

$$L\{e^{5t} + \sin(5t)\} = L\{e^{5t}\} + L\{\sin(5t)\} = \frac{1}{s-5} + \frac{5}{s^2+25}, s > 5.$$

Find $L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\}$.

By linearity (6) we have

$$L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\}$$

$$= L\{e^{2t}\} + 4L\{t^3\} - 2L\{\sin 3t\} + 3L\{\cos 3t\}$$

$$= \frac{1}{s-2} + 4\times\frac{3!}{s^4} - 2\times\frac{3}{s^2+3^2} + 3\times\frac{s}{s^2+3^2}$$

$$= \frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+9}, \quad s > 2.$$

Examples on Linearity

Find $L\{7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t\}$.

By linearity (6) we have

$$L\{7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t\}$$
$$= 7L\{e^{2t}\} + 9L\{e^{-2t}\} + 5L\{\cos t\} + 7L\{t^3\} + 5L\{\sin 3t\}$$
$$= 7\frac{1}{s-2} + 9\times\frac{1}{s+2} + 5\times\frac{s}{s^2+1^2} + 7\times\frac{3!}{s^4} + 5\times\frac{3}{s^2+3^2}$$
$$= \frac{7}{s-2} + \frac{9}{s+2} + \frac{5s}{s^2+1} + \frac{42}{s^4} + \frac{15}{s^2+9}, \quad s > 2.$$

Examples on Linearity

Find $L\{\sinh(at)\} \& L\{\cosh(at)\}$ where a is a constant.

We know that $\sinh(at) = \frac{1}{2}\left(e^{at} - e^{-at}\right)$, $\cosh(at) = \frac{1}{2}\left(e^{at} + e^{-at}\right)$. By linearity (6) we have

$$L\{\sinh(at)\} = L\left\{\frac{1}{2}\left(e^{at} - e^{-at}\right)\right\}$$

$$= \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{1}{2}\left(\frac{s+a-s+a}{(s+a)(s-a)}\right)$$

$$= \frac{a}{s^2 - a^2}, \quad s > |a|.$$

$$L\{\sinh(at)\} = \frac{a}{s^2 - a^2}; \quad L\{\cosh(at)\} = \frac{s}{s^2 - a^2}, \quad s > |a|.$$

Examples on Linearity

Find $L\{(5e^{2t}-3)^2\}$.

$$L\{(5e^{2t}-3)^2\} = L\{25e^{4t}-30e^{2t}+9\}$$
$$= 25L\{e^{4t}\} - 30L\{e^{2t}\} + L\{9\}$$
$$= 25\frac{1}{s-4} - 30\frac{1}{s-2} + 9\frac{1}{s}$$
$$= \frac{25}{s-4} - \frac{30}{s-2} + \frac{9}{s}, \quad s > 4.$$

Examples on Linearity

Find $L\{(5e^{2t} - 3)^2\}$.

$$L\{(5e^{2t} - 3)^2\} = L\{25e^{4t} - 30e^{2t} + 9\}$$

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$$= 25\frac{1}{s-4} - 30\frac{1}{s-2} + 9\frac{1}{s}$$

$$= \frac{25}{s-4} - \frac{30}{s-2} + \frac{9}{s}, \quad s > 4.$$

Examples on Linearity

Find $L\{\sin^2 at\}$.

Since $\sin^2(at) = \frac{1}{2}\left(1 - \cos(2at)\right)$. Then we have

$$L\{\sin^2(at)\} = \frac{1}{2}L\{1 - \cos(2at)\}$$

$$= \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos(2at)\}$$

$$= \frac{1}{2} \times \frac{1}{s} - \frac{1}{2} \times \frac{s}{(s)^2 + (2a)^2}$$

$$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2 + 4a^2} = \frac{4a^2}{2s(s^2 + 4a^2)}.$$

(Piecewise continuous function)

Definition

A function f is piecewise continuous on the interval $[a, b]$ if

1. The interval $[a, b]$ can be broken into a finite number of subintervals $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ such that f is continuous in each subinterval (t_i, t_{i+1}), for $i = 0, 1, 2, \cdots n - 1$
2. The function f has jump discontinuity at t_i

$$\lim_{t \to t_i^+} f(t) < \infty, i = 0, 1, 2, \cdots n - 1$$

and

$$\lim_{t \to t_i^-} f(t) < \infty, i = 0, 1, 2, \cdots n.$$

Note: A function is piecewise continuous on $[0, \infty)$ if it is piecewise continuous in $[0, A]$ for all $A > 0$.

Examples

1. The function

$$f(t) = \begin{cases} t, & 0 \le t < 1; \\ t^2, & 1 \le t < 2; \\ t+1, & 2 \le t \le 3. \end{cases}$$

   is piecewise continuous on $[0, 3]$

2. The functions $f(t) = [x], x - [x]$, ets are piecewise continuous on $(0, \infty)$.

3. The function

$$\begin{cases} \frac{1}{t-3}, & 0 \le t < 3; \\ t+1, & 3 \le t \le 4; \end{cases}$$

   is not piecewise continuous on $[0, 4]$.

Exponential order

Definition

A function $f(t)$ is said to be of exponential order $a(>0)$ on $0 \leq t < \infty$ if there exists a positive constant K such that for all $t > c$

$$|f(t)| \leq Ke^{at}$$

Or, equivalently,

$$\lim_{t\to\infty} \frac{f(t)}{e^{at}} = 0 \tag{7}$$

and we write this symbolically as

$$f(t) = O(e^{at}) \quad \text{as } t \to \infty$$

Existence Conditions for the Laplace Transform

Theorem

If a function $f(t)$ is continuous or piecewise continuous in every finite interval in $[0, \infty)$, and of exponential order e^{at}, then the Laplace transform of $f(t)$ exists for all s provided $s > a$ and

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty} L\ \{f(t)\} = 0.$$

Proof.

Since, $f(t)$ is piecewise continuous and of exponential order e^{at} on $[0, \infty)$, then there exists s a positive constant K such that for all $t > c$

$$|f(t)| \leq Ke^{at}$$

The integral in the definition of $F(s)$ can be split into two integrals as follows

$$\int_0^\infty e^{-st} f(t)dt = \int_0^c e^{-st} f(t)dt + \int_c^\infty e^{-st} f(t)dt$$

Since $f(t)$ is piecewise continuous in $0 \le t \le c$ it is bounded there. By taking $A = \max\{f(t) : 0 \le t \le c\}$ we have

$$\int_0^c e^{-st} f(t)\, dt \le A \int_0^c e^{-st} dt$$

$$= A \left[\frac{e^{-st}}{-s}\right]_0^c$$

$$= A \left[\frac{1}{s} - \frac{e^{-sc}}{s}\right] < \infty \quad (8)$$

Which implies that the integral $\int_0^c e^{-st} f(t)dt$ converges. The integral

$$\int_c^\infty e^{-st} f(t)dt \le K \int_c^\infty e^{-st} e^{at} dt$$

$$= K \int_c^\infty e^{-t(s-a)} dt$$

$$\int_c^{\infty} e^{-st} f(t)dt \leq \frac{K}{s-a} < \infty \quad s > a. \tag{9}$$

Thus the integral $\int_c^{\infty} e^{-st} f(t)dt$ converges. Hence, the Laplace transform $L\{f(t)\}$ converges.

Note: The condition in the above theorem is not necessary.

Example-I: consider $f(t) = \frac{1}{\sqrt{t}}$, which is not piecewise continuous in $[0, \infty)$. But

$$L\left(\frac{1}{\sqrt{t}}\right) = \int_0^\infty \frac{e^{-st}}{\sqrt{t}}\,dt = \frac{1}{\sqrt{s}}\int_0^\infty e^{-u}u^{1/2-1}\,du = \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad s > 0.$$

Example-II: Consider

$$f(t) = 2te^{t^2}\cos\left(e^{t^2}\right)$$

Then $f(t)$ is continuous on $[0,\infty)$ but not of exponential order. However, the Laplace transform of $f(t)$,

$$L\left\{2te^{t^2}\cos\left(e^{t^2}\right)\right\} = \int_0^\infty e^{-st}\,2te^{t^2}\cos\left(e^{t^2}\right)dt$$

exists, since integration by parts yields

$$L\left\{2te^{t^2}\cos\left(e^{t^2}\right)\right\} = \left[e^{-st}\sin\left(e^{t^2}\right)\right]_0^\infty + s\int_0^\infty e^{-st}\sin\left(e^{t^2}\right)dt$$

$$= -\sin(1) + sL\left\{\sin\left(e^{t^2}\right)\right\}$$

and the latter Laplace transform exists.

By equation (8) and (9), it is easy to see that

$$\lim_{s\to\infty} F(s) \leq \lim_{s\to\infty} A\left(\frac{1}{s} - \frac{e^{-sc}}{s} + \frac{K}{s-a}\right) = 0 \quad (10)$$

Note that:

- Any function $F(s)$ without this behaviour (10) can not be Laplace transform of a certain function. For example, $F(s) = s$, $F(s) = \sin s$, $F(s) = \frac{s}{1+s}$ are not Laplace transform of any function.

Problems

Find $L\{F(t)\}$, where $F(t) = \begin{cases} 0, & 0 < t < 1; \\ t, & 1 < t < 2; \\ 0, & t > 2. \end{cases}$

By definition we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^1 e^{-st} F(t)\,dt + \int_1^2 e^{-st} F(t)\,dt + \int_2^\infty e^{-st} F(t)\,dt$$

$$= \int_0^1 e^{-st}(0)\,dt + \int_1^2 e^{-st}\,t\,dt + \int_2^\infty e^{-st}(0)\,dt = \int_1^2 e^{-st}\,t\,dt$$

$$= \left[t \times \frac{e^{-st}}{-s} - 1 \times \frac{e^{-st}}{s^2} \right]_1^2 = \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_1^2$$

$$= \frac{-2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} = -e^{-2s}\left(\frac{2}{s} + \frac{1}{s^2}\right) + e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$$

Problems

Find $L\{F(t)\}$, where $F(t) = \begin{cases} (t-1)^2, & t > 1; \\ 0, & 0 < t < 1. \end{cases}$ By definition we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = \int_0^1 e^{-st}(0)dt + \int_1^\infty e^{-st}(t-1)^2 dt$$

$$= \left[(t-1)^2 \frac{e^{-st}}{-s} - (2t-2)\frac{e^{-st}}{s^2} + (2)\frac{e^{-st}}{-s^3}\right]_1^\infty$$

$$= \frac{2e^{-s}}{s^3}$$

Gamma Function

The classical Euler's gamma function is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad x > 0. \tag{11}$$

Properties

1. $\Gamma(1) = 1$
2. $\Gamma(n + 1) =$
3. $n!$
4. $\Gamma(x + 1) = x\Gamma(x)$ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$L\{\sqrt{t}\}$

Solution:

By definition we have

$$L\{\sqrt{t}\} = \int_0^\infty e^{-st}\sqrt{t}\,dt$$

Let $st = u$ then we have $sdt = du$ and $t = u/s$ we have

$$= \int_0^\infty e^{-u}\sqrt{\frac{u}{s}}\,\frac{du}{s} = \frac{1}{s\sqrt{s}}\int_0^\infty e^{-u}\sqrt{u}\,du$$

$$= \frac{1}{s\sqrt{s}}\int_0^\infty e^{-u}u^{\frac{1}{2}+1-1}\,du$$

$$= \frac{1}{s\sqrt{s}}\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{s\sqrt{s}}\times\frac{1}{2}\times\Gamma(1/2)$$

$$= \frac{1}{s\sqrt{s}}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2s\sqrt{s}}$$

$L\{t^{\alpha}\}, \alpha \in \mathbb{R}$

Solution:

By definition we have

$$L\{t^{\alpha}\} = \int_0^{\infty} e^{-st} t^{\alpha}\, dt$$

Let $st = u$ then we have $sdt = du$ and $t = u/s$ we have

$$= \int_0^{\infty} e^{-u} \left(\frac{u}{s}\right)^{\alpha} \frac{du}{s} = \frac{1}{s^{\alpha+1}} \int_0^{\infty} e^{-u} u^{\alpha} du$$

$$= \frac{1}{s^{\alpha}} \int_0^{\infty} e^{-u} u^{\alpha+1-1} du$$

$$= \frac{1}{s^{\alpha+1}} \Gamma(\alpha + 1) \quad \text{using (11)}$$

$$\boxed{L\{t^{\alpha}\} = \frac{\Gamma(a+1)}{s^{\alpha+1}}}$$

Problems

Find the Laplace Transform of the following functions

1. $F(t) = \begin{cases} 4, & 0 < t < 1; \\ 3, & t > 1; \end{cases}$
2. $(\sin t - \cos t)^2$
3. $|t-1| + |t+1|, t > 0$
4. $\sin 2t \cos 3t$
5. $\sin^3 t$
6. $\cos^3 2t$
7. $\cosh^2 2t$
8. $\sin(\sqrt{t}$
9. $)^{2t}$
10. $(t+1)^2$ e $\sin 3t$
11. $\sin 2t \sin 3t$
12. $\cos 2t \cos 3t$

Topics

1. Heaviside's First Shifting Theorem Second
2. Shifting Theorem
3. Change of Scale Property
4. Laplace Transform of the derivative of $f(t)$
5. Initial Value theorem and Final Value theorem

Heaviside's First Shifting Theorem

Theorem

If $L\{f(t)\} = F(s)$,
then

$$L\{e^{at} f(t)\} = F(s-a) \quad (12)$$

where a *is a real constant.*

Proof.

By definition we have

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt \quad (13)$$

Using the change of variable $u = s - a$ the equation (13) reduces to

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)dt = \int_0^\infty e^{-ut} f(t)dt = F(u) = F(s-a)$$

Examples

From the equation (12), we have

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

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$$L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}$$

$$L\{\sin(bt)\} = \frac{b}{s^2+b^2}$$

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$$L\{\sinh(bt)\} = \frac{b}{s^2-b^2} \Rightarrow L\{e^{at}\sinh(bt)\} = \frac{b}{(s-a)^2-b^2}$$

$$L\{\cosh(bt)\} = \frac{s}{s^2-b^2} \Rightarrow L\{e^{at}\cosh(bt)\} = \frac{s-a}{(s-a)^2-b^2}$$

Find the Laplace Transform of $e^{-t}\cos 2t$

Solution: First we find Laplace transform of $\cos 2t$.

$$L\{\cos 2t\} = \frac{s}{s^2+2^2}$$

$$\frac{s}{s^2+4} = F(s)$$

by first shifting theorem we have

$$L\{e^{-t}\cos 2t\} == \frac{s-(-1)}{(s-(-1))^2+4}$$

$$= \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

Example

$$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$$

Solution: First we find the Laplace transform of $2\cos 5t - 3\sin 5t$.

$$L\{2\cos 5t - 3\sin 5t\} = 2L\{\cos 5t\} - 3L\{\sin 5t\} \quad \text{linearity}$$

$$= 2\times\frac{s}{s^2+5^2} - 3\times\frac{5}{s^2+5^2}$$

$$= \frac{2s}{s^2+25} - \frac{15}{s^2+25} = \frac{2s-15}{s^2+25} = F(s)$$

by first shifting theorem we have

$$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = F(s+3)$$

$$= \frac{2(s+3)-15}{(s+3)^2+25} = \frac{2s+6-15}{s^2+6s+9+25}$$

$$= \frac{2s-9}{s^2+6s+34}$$

Example

Find $L\{e^{3t}\sin^2 t\}$

Solution: First, we find the Laplace transform of $\sin^2 t$. By definition

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\} \quad \text{since, } \sin^2 t = \frac{1}{2}[1-\cos 2t]$$

$$= \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 2t\}$$

$$= \frac{1}{2s} - \frac{1}{2s} \times \frac{s}{s^2+2^2} = \frac{1}{2s} - \frac{s}{2(s^2+4)}$$

$$= \frac{2s^2+8-2s^2}{4s(s^2+4)} = \frac{2}{s(s^2+4)} = F(s)$$

by first shifting theorem we have

$$L\{e^{3t}\sin^2 t\} = \frac{2}{(s-3)((s-3)^2+4)} = \frac{2}{(s-3)(s^2-6s+13)}$$

Example

Find $L\{\sinh at \sin at\}$

Solution: $\sinh at = \frac{1}{2}[e^{at} - e^{-at}]$ and $\sinh at \sin at = \frac{1}{2}[e^{at} \sin at - e^{-at} \sin at]$.

$$L\{\sinh at \sin at\} = L\left\{\frac{1}{2}\left(e^{at}\sin at - e^{-at}\sin at\right)\right\}$$

$$= \frac{1}{2}L\{e^{at}\sin at\} - \frac{1}{2}L\{e^{-at}\sin at\} \quad \text{linearity}$$

$$= \frac{1}{2}\frac{a}{(s-a)^2+a^2} - \frac{1}{2}\frac{a}{(s+a)^2+a^2}; \text{shifting theorem}$$

$$= \frac{a}{2}\frac{(s+a)^2+a^2-(s-a)^2-a^2}{\left[(s+a)^2+a^2\right]\times\left[(s-a)^2+a^2\right]}$$

$$= \frac{a}{2}\frac{4sa}{(s^2+2a^2-2as)\times(s^2+2a^2+2as)}$$

$$= \frac{2sa^2}{(s^2+2a^2)^2-(2as)^2} = \frac{2sa^2}{s^4+4a^4}$$

Example

Find $L\{\cosh ta \cos at\}$

Solution: We know that $L\{\cos at\} = \frac{s}{s^2+a^2}$ and $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$.

Now, we find

$$L\{\cosh at \cos at\} = \frac{1}{2}L\{e^{at}\cos at + e^{-at}\cos at\}$$

$$= \frac{1}{2}L\{e^{at}\cos at\} + \frac{1}{2}L\{e^{-at}\cos at\} \quad \text{linearity}$$

$$\text{shifting thm} = \frac{1}{2}F(s-a) + \frac{1}{2}F(s+a); \text{ where } F(s) = \frac{s}{s^2+a^2}$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)[(s-a)^2+a^2] + (s+a)[(s-a)^2+a^2]}{(s^2+2a^2-2as)(s^2+2a^2-2as)}\right]$$

$$= \frac{s^3}{s^4+4a^4}$$

Example

Find $L\ \{(t+3)^3 e^t\}$

Solution: First we find $L\ \{(t+1)^2\}$.

$$L\ \{(t+3)^2\} = L\ \{t^2+6t+9\}$$
$$= L\ \{t^2\} + 6L\ \{t\} + L\ \{9\} \quad \text{linearity}$$
$$= \frac{2}{s3} + \frac{6}{s2} + \frac{9}{s} = F(s)$$

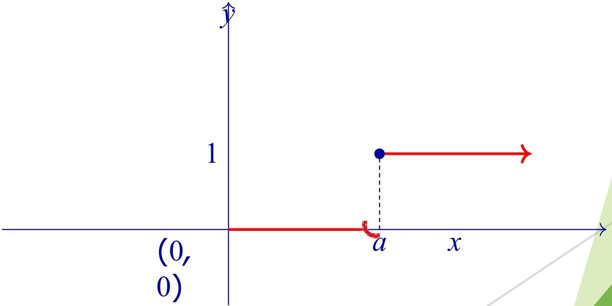
by first shifting theorem we have

$$L\ \{(t+3)^2 e^t\} = F(s-1)$$
$$= \frac{2}{(s-1)^3} + \frac{6}{(s-1)^2} + \frac{9}{s-1}$$

Heaviside unit step function

Definition: Let $a \geq 0$. The Heaviside unit function $H(t-a) = U(t-a)$ is defined by

$$H(t-a) = U(t-a) = \begin{cases} 1, & \text{if } t \geq a; \\ 0, & \text{if } 0 \leq t < a. \end{cases} \quad (14)$$



$L\{U(t-a)\}$

Find $L\{U(t-a)\}$

Sol: By definition

$$L\{U(t-a)\} = \int_0^\infty e^{-st}U(t-a)dt$$

$$= \int_0^a e^{-st}U(t-a)dt + \int_a^\infty e^{-st}U(t-a)dt$$

$$= \int_a^\infty e^{-st}U(t-a)dt = \int_a^\infty e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s} \quad s > 0.$$

$$\boxed{L\{U(t-a)\} = \frac{e^{-sa}}{s}, \quad s > 0.} \qquad (15)$$

The unit step function can be used to express piecewise functions.

Example: Express the piecewise defined function in terms of the unit step function $f(t) = \begin{cases} 4, & \text{if } 0 \le t < 9 \\ 6, & \text{if } t \ge 9. \end{cases}$

Look at the function

$$4 + (6 - 4)U(t - 9)$$

If $0 \le t < 9$, then $U(t - 9) = 0$. Then

$$4 + (6 - 4)U(t - 9) = 4 + (6 - 4)0 = 4$$

and if $t \ge 9$, then $U(t - 9) = 1$. Then

$$4 + (6 - 4)U(t - 9) = 4 + (6 - 4)1 = 6$$

Thus, we see that

$$f(t) = 4 + (6 - 4)U(t - 9) = 4 + 2U(t - 9)$$

Consider the piecewise defined function

$$f(t) = \begin{cases} t, & \text{if } 0 \le t < 2 \\ t^2, & \text{if } 2 \le t < 6 \\ t^3, & \text{if } t \ge 6. \end{cases}$$

We can express f in terms of unit functions as.

$$\boxed{f(t) = t + (t^2 - t)U(t-2) + (t^3 - t^2)U(t-6)}$$

Note that:

$$g(t) = \begin{cases} 0, & \text{if } 0 \le t < a \\ f(t), & \text{if } t \ge a. \end{cases} \rightarrow \boxed{g(t) = f(t)U(t-a)}$$

Example

Find $L\ \{e^{3t}H(t-2)\}$

Solution: We know that $F(s) = L\ \{H(t-2)\} = \frac{e^{-2s}}{s}$. By first shifting theorem we have

$$L\ \{e^{3t}H(t-2)\} = F(s-3)$$

$$= \frac{e^{-2(s-3)}}{s-3}$$

Second Shifting Theorem

Theorem

Let $a \geq 0$ *and* $L\{f(t)\} = F(s)$. *Then*

$$L\{f(t-a)U(t-a)\} = e^{-as}F(s) \quad (16)$$

Or, equivalently,

$$L\{f(t)U(t-a)\} = e^{-as}F(s+a) \quad (17)$$

where $U(t-a)$ *is the Heaviside unit step function defined by* (14).

or equivalently

If $L\{f(t)\} = F(s)$ *and* $g(t) = \begin{cases} f(t-a), & t > a; \\ 0, & 0 < t < a. \end{cases}$ *then* $L\{g(t)\} = e^{-as}F(s)$.

Proof.

From the definition

$$L\{f(t-a)U(t-a)\} = \int_0^\infty e^{-st} f(t-a)U(t-a)dt$$

$$= \int_0^a e^{-st} f(t-a)U(t-a)dt + \int_a^\infty e^{-st} f(t-a)U(t-a)dt$$

$$= \int_a^\infty e^{-st} f(t-a)dt$$

which is, by putting $t - a = u$, we get

$$= \int_a^\infty e^{-st} f(t-a)dt = \int_0^\infty e^{-s(a+u)} f(u)du$$

$$= e^{-as}\int_0^\infty e^{-su} f(u)du = \bar{e}^{as} F(s).$$

Problem1: Find the Laplace transform of $(t-2)^3 H(t-2)$

Solution: compare the given function with $f(t-a)H(t-a)$, we have $a=2$ and $f(t)=t^3$. We know that $L\{f(t)\} = \frac{6}{s^4} = F(s)$. By second shifting theorem we have

$$L\{(t-2)^3 H(t-2)\} = e^{-as}F(s)$$

$$= e^{-2s} \times \frac{6}{s^4} = \frac{6e^{-2s}}{s^4}.$$

$$\boxed{L\{(t-2)^3 H(t-2)\} = \frac{6e^{-2s}}{s^4}.}$$

Problem2: Find the Laplace transform of

$$f(t) = \begin{cases} \cos\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3}; \\ 0, & 0 < t < \frac{\pi}{3}. \end{cases}$$

Solution: First we express the given function in-terms of unit step function. $f(t) = \cos\left(t - \frac{\pi}{3}\right) H\left(t - \frac{\pi}{3}\right)$ and compare the given function with $f(t-a)H(t-a)$, we have $a = \frac{\pi}{3}$ and $f(t) = \cos t$. We know that $L\{f(t)\} = \frac{s}{s^2+1} = F(s)$. By second shifting theorem we have

$$L\left\{\cos\left(t - \frac{\pi}{3}\right) H\left(t - \frac{\pi}{3}\right)\right\} = e^{-as}F(s)$$

$$= e^{-\frac{\pi s}{3}} \times \frac{s}{s^2+1} = \frac{se^{-\frac{\pi s}{3}}}{s^2+1}.$$

$$L\{f(t)\} = L\left\{\cos\left(t - \frac{\pi}{3}\right) H\left(t - \frac{\pi}{3}\right)\right\} = \frac{se^{-\frac{\pi s}{3}}}{s^2+1}.$$

Problem 3: Find the Laplace transform of

$$f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3}; \\ 0, & 0 < t < \frac{2\pi}{3}. \end{cases}$$

Solution: First we express the given function in-terms of unit step function. $f(t) = \cos\left(t - \frac{2\pi}{3}\right) H\left(t - \frac{2\pi}{3}\right)$ and compare the given function with $f(t - a)H(t - a)$, we have $a = \frac{2\pi}{3}$ and $f(t) = \cos t$. We know that $L\{f(t)\} = \frac{s}{s^2+1} = F(s)$. By second shifting theorem we have

$$L\left[\cos\left(t - \frac{2\pi}{3}\right) H\left(t - \frac{2\pi}{3}\right)\right] = e^{-as} F(s)$$

$$= e^{-\frac{2\pi s}{3}} \times \frac{s}{s^2+1} = \frac{s e^{-\frac{2\pi s}{3}}}{s^2+1}.$$

$$L\{f(t)\} = L\left[\cos\left(t - \frac{2\pi}{3}\right) H\left(t - \frac{2\pi}{3}\right)\right] = \frac{s\, e^{-\frac{2\pi s}{3}}}{s^2+1}.$$

Problem 4: Find the Laplace transform of

$$f(t) = \begin{cases} \cos\left(t - \frac{3\pi}{4}\right), & t > \frac{3\pi}{4}; \\ 0, & 0 < t < \frac{2\pi}{3}. \end{cases}$$

Solution: First we express the given function in-terms of unit step function. $f(t) = \cos\left(t - \frac{3\pi}{4}\right) H\left(t - \frac{3\pi}{4}\right)$ and compare the given function with $f(t-a)H(t-a)$, we have $a = \frac{3\pi}{4}$ and $f(t) = \cos t$. We know that $L\{f(t)\} = \frac{s}{s^2+1} = F(s)$. By second shifting theorem we have

$$L\left[\cos\left(t - \frac{3\pi}{4}\right) H\left(t - \frac{3\pi}{4}\right)\right] = e^{-as} F(s)$$

$$= e^{-\frac{3\pi s}{4}} \times \frac{s}{s^2+1} = \frac{s e^{-\frac{3\pi s}{4}}}{s^2+1}.$$

$$\boxed{L\{f(t)\} = L\left[\cos\left(t - \frac{3\pi}{4}\right) H\left(t - \frac{3\pi}{4}\right)\right] = \frac{s e^{-\frac{3\pi s}{4}}}{s^2+1}.}$$

Problem 5: Find the Laplace transform of

$$f(t) = \begin{cases} \sin\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3}; \\ 0, & 0 < t < \frac{\pi}{3}. \end{cases}$$

Solution: First we express the given function in-terms of unit step function. $f(t) = \cos\left(t - \frac{\pi}{3}\right) H\left(t - \frac{\pi}{3}\right)$ and compare the given function with $f(t-a)H(t-a)$, we have $a = \frac{\pi}{3}$ and $f(t) = \sin t$. We know that $L\{f(t)\} = \frac{1}{s^2+1} = F(s)$. By second shifting theorem we have

$$L\left\{\cos\left(t - \frac{\pi}{3}\right) H\left(t - \frac{\pi}{3}\right)\right\} = e^{-as} F(s)$$

$$= e^{-\frac{\pi s}{3}} \times \frac{1}{s^2+1} = \frac{e^{-\frac{\pi s}{3}}}{s^2+1}.$$

$$\boxed{L\{f(t)\} = L\left\{\cos\left(t - \frac{\pi}{3}\right) H\left(t - \frac{\pi}{3}\right)\right\} = \frac{e^{-\frac{\pi s}{3}}}{s^2+1}.}$$

Problem 6: Find the Laplace transform of $(t-1)^3H(t-1)$

Solution: compare the given function with $f(t-a)H(t-a)$, we have $a = 1$ and $f(t) = t^3$. We know that $L\{f(t)\} = \frac{6}{s^4} = F(s)$. By second shifting theorem we have

$$L\{(t-1)^3H(t-1)\} = e^{-as}F(s)$$

$$= e^{-s} \times \frac{6}{s^4} = \frac{6e^{-s}}{s^4}.$$

$$\boxed{L\{(t-1)^3H(t-1)\} = \frac{6e^{-s}}{s^4}.}$$

Problem 7: Find the Laplace transform of $e^{-2t}(1 - H(t-1))$

Solution: Given function $e^{-2t}[(1 - H(t-1))$ can be expressed as $e^{-2t} - e^{-2t}H(t-1)$. compare the second part of the function with $f(t-a)H(t-a)$, we have $a = 1$ and $f(t) = e^{-2t}$. We know that $L\{f(t)\} = \frac{1}{s+2} = F(s)$. By second shifting theorem we have

$$L\{e^{-2t}(1 - H(t-1))\} = L\{e^{-2t} - e^{-2t}H(t-1)\}$$

$$= L\{e^{-2t}\} - L\{e^{-2t}H(t-1)\} \quad \text{linearity}$$

$$= \frac{1}{s+2} - L\left\{e^{-2(t-1+1)}H(t-1)\right\},$$

$$= \frac{1}{s+2} - e^{-2}L\left\{e^{-2(t-1)}H(t-1)\right\},$$

$$= \frac{1}{s+2} - e^{-2}e^{-s} \times \frac{1}{s+2} \quad \text{II shifting theorem}$$

$$= \frac{1}{s+2} - \frac{e^{-(s+2)}}{s+2}$$

Problem 8: Find the Laplace transform of

$$g(t) = \begin{cases} (t-1)^2, & t > 1; \\ 0, & 0 < t < 1. \end{cases}$$

Solution: First we express the given function in-terms of unit step function.
$g(t) = (t-1)^2 H(t-1)$ and compare the given function with $f(t-a)H(t-a)$, we have $a = 1$ and $f(t) = t^2$. We know that $L\{f(t)\} = L\{t^2\} = \frac{2}{s^3} = F(s)$.
By second shifting theorem we have

$$L\{(t-1)^2 H(t-1)\} = e^{-as}F(s)$$

$$= e^{-s} \times \frac{2}{s^3} = \frac{2e^{-s}}{s^3}.$$

$$\boxed{L\{g(t)\} = L\{(t-1)^2 H(t-1)\} = \frac{2e^{-s}}{s^3}.}$$

Solution 1 of the above problem can be found in Unit-I !

Change of Scale Property

Theorem

Change of Scale Property: If $L\{f(t)\} = F(s)$*, then*

$$L\{f(at)\} = \frac{1}{a}F(s/a) \tag{18}$$

Proof.

From the definition we have

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt$$

$$= \int_0^\infty e^{-su/a} f(u)du/a \quad \text{put } at = u, \text{ then } dt = du/a$$

$$= \frac{1}{a}\int_0^\infty e^{-u(s/a)} f(u)du = \frac{1}{a}F(s/a)$$

Problem : $L\{f(t)\} = \frac{9s^2 - 12s + 15}{(s-1)^3}$, find $L\{f(3t)\}$.

Solution:

Given that $L\{f(t)\} = \frac{9s^2 - 12s + 15}{(s-1)^3} = F(s)$

by change of scale property we have

$$L\{f(3t)\} = \frac{1}{3}F(s/3) = \frac{1}{3}\frac{9(s/3)^2 - 12(s/3) + 15}{(s/3-1)^2}$$

$$= \frac{27}{9}\frac{s^2 - 4s + 15}{(s-3)^3}$$

$$= \frac{9(s^2 - 4s + 15)}{(s-3)^3}$$

Problem: $L\{f(t)\} = \frac{1}{s}e^{\frac{-1}{s}}$, show $L\{e^{-t}f(3t)\} = \frac{e^{\frac{-3}{s+1}}}{s+1}$.

Solution:

Given that $L\{f(t)\} = \frac{e^{\frac{-1}{s}}}{s} = F(s)$

by change of scale property we have

$$L\{f(3t)\} = \frac{1}{3}F(s/3) = \frac{1}{3} \times \frac{3}{s} \times e^{\frac{-3}{s}} = \frac{e^{\frac{-3}{s}}}{s} = G(s)$$

applying first shifting theorem we obtain

$$L\{e^{-t}f(3t)\} = G(s+1) = \frac{e^{\frac{-3}{s+1}}}{s+1}$$

Problem: $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$ find $L\left\{\frac{\sin at}{t}\right\}$.

Solution:

$$\text{Given that } L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s} = F(s)$$

by change of scale property we have

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}F(s/a) = \frac{1}{a}\times\tan^{-1}\frac{a}{s}$$

$$\frac{1}{a}L\left\{\frac{\sin at}{t}\right\} = \frac{1}{a}\tan^{-1}\frac{a}{s}$$

$$L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\frac{a}{s}$$

Laplace Transform of Derivatives-I

Theorem

If $L\ \{f(t)\} = F(s)$,
then

$$L\ \{f'(t)\} = sF(s) - f(0) \quad (19)$$

From the definition of the Laplace transform we have

$$L\ [f'(t)] = \int_0^{\infty} e^{-st} f'(t)dt$$

Using the integration by parts we have

$$L\ [f'(t)] = \left[e^{-st} f(t)\right]_0^{\infty} - \int_0^{\infty} e^{-st}(-s) f(t)dt = 0 - f(0) + s\int_0^{\infty} e^{-st} f(t)dt$$

Assuming that $\lim_{s\to\infty} e^{-st} f(t) = 0$

$$= sL\ (f(t)) - f(0) = sF(s) - f(0) = L\ \{f'(t)\} = sF(s) - f(0)$$

Laplace Transform of Derivatives-II

Theorem

If $L\{f(t)\} = F(s)$,
then

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0) \quad (20)$$

$$L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0) \quad (21)$$

$$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \quad (22)$$

If $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s\sqrt{s}}$ show that $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

Solution: Let $f(t) = 2\sqrt{\frac{t}{\pi}}$ then we have $f(0) = 0$ and

$$f'(t) = \frac{d}{dt}2\sqrt{\frac{t}{\pi}} = \frac{2}{\sqrt{\pi}}\frac{d}{dt}\sqrt{t}$$

$$= \frac{2}{\sqrt{\pi}} \times \frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{\pi}} \times \frac{1}{\sqrt{t}}$$

$$f'(t) = \frac{1}{\sqrt{\pi t}}$$

Now find

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = s \times \frac{1}{s\sqrt{s}} - 0$$

The Initial Value Theorem

Theorem

If $L\{f(t)\} = F(s)$, *and if* (t) *and its derivatives exist as* $t \to 0$

$$\lim_{s\to\infty}(sF(s)) = \lim_{t\to 0} f(t) = f(0)$$

$$\lim_{s\to\infty}(s^2 F(s) - sf(0)) = \lim_{t\to 0} f'(t) = f'(0)$$

Proof.

Since Laplace integral is converges with respect to the parameter s. Hence, it is permissible to take the limit $s \to \infty$ under the sign of integration so that

$$\lim_{s\to\infty} F(s) = \int_0^\infty \lim_{s\to\infty} e^{-st} f(t)dt = 0$$

Next, we use the same argument to obtain

$$\lim_{s\to\infty} L\{f'(t)\} = \int_0^\infty \lim_{s\to\infty} e^{-st} f'(t)dt = 0$$

Then it follows from result (19) that

$$\lim_{s\to\infty} L\{f'(t)\} = \lim_{s\to\infty}[sF(s) - f(0)] = 0$$

and hence, we obtain

$$\boxed{\lim_{s\to\infty}[sF(s)] = f(0) = \lim_{t\to 0} f(t)}$$

Problem: Find $f(0)$ and $f'(0)$ when $F(s) = \frac{1}{(s^2+25)}$ and $F(s) = \frac{2s}{s^2-2s+25}$.

Solution:

$$f(0) = \lim_{s\to\infty}[sF(s)] = \lim_{s\to\infty} \frac{s}{s(s^2+25)} = \lim_{s\to\infty} \frac{1}{(s^2+25)} = 0$$

$$f'(0) = \lim_{s\to\infty}[s^2 F(s) - s\,f(0)] = \lim_{s\to\infty} \frac{s}{s(s^2-2s+25)} - 0$$

$$= \lim_{s\to\infty} \frac{1}{s^2-2s+25} = 0$$

and

$$f(0) = \lim_{s\to\infty}[sF(s)] = \lim_{s\to\infty} \frac{2s^2}{s^2-2s+25} = 2$$

$$f'(0) = \lim_{s\to\infty}[s^2F(s) - s\,f(0)] = \lim_{s\to\infty} \frac{2s^3}{s^2-2s+25} - 2s = 4$$

The Final Value Theorem

Theorem

If $L\{f(t)\} = F(s)$,
then

$$\lim_{s\to 0}(F(s)) = \int_0^\infty f(t)dt$$

$$\lim_{s\to 0}(sF(s)) = \lim_{t\to\infty} f(t)$$

Proof

Proof.

We use the same argument as employed in previous theorem and find

$$\lim_{s\to 0}(F(s)) = \int_0^{\infty} \lim_{s\to 0} e^{-st} f(t)dt = \int_0^{\infty} f(t)dt$$

As before, we can use result (19) to obtain

$$\lim_{s\to 0} L\{f'(t)\} = \lim_{s\to 0}[sF(s) - f(0)] = \int_0^{\infty} \lim_{s\to 0} e^{-st} f'(t)dt$$

$$= \int_0^{\infty} f'(t)dt = f(\infty) - f(0) = \lim_{t\to\infty}(f(t) - f(0))$$

Thus, it follows immediately that

$$\boxed{\lim_{s\to 0}[sF(s)] = \lim_{t\to\infty} f(t) = f(\infty)}$$

Problems: Show that $\int_0^\infty te^{-3t}\sin t\,dt = \frac{3}{50}$.

Solution: We know that $L\{\sin t\} = \frac{1}{1+s^2}$. In integral from

$$L\{\sin t\} = \int_0^\infty e^{-st}\sin t\,dt = \frac{1}{1+s^2}$$

$$\int_0^\infty e^{-st}\sin t\,dt = \frac{1}{1+s^2}$$

differentiate with respect to s both sides we get

$$\frac{d}{ds}\int_0^\infty e^{-st}\sin t\,dt = \frac{d}{ds}\frac{1}{1+s^2} \Rightarrow \int_0^\infty e^{-st}\sin t(-t)dt = \frac{-2s}{(1+s^2)^2}$$

$$\int_0^\infty te^{-st}\sin t\,dt = \frac{2s}{(1+s^2)^2}$$ taking limit $s \to 3$ we get

$$\int_0^\infty te^{-3t}\sin t\,dt = \frac{2\times 3}{(1+3^2)^2} = \frac{6}{100} = \frac{3}{50}$$

Question

since, $L\{t\sin t\} = \int_0^\infty t\sin t\, e^{-st}dt = \frac{2s}{(1+s^2)^2}$ Now

$$\int_0^\infty t\sin t\, e^{-3t}dt = \frac{3}{50} \quad \text{as taking } s \to 3$$

$$\int_0^\infty t\sin t\, e^{-2t}dt = \frac{4}{25} \quad \text{as taking } s \to 2$$

$$\int_0^\infty t\sin t\, e^{-4t}dt = \frac{8}{289} \quad \text{as taking } s \to 4$$

$$\int_0^\infty t\sin t\, e^{2t}dt = \frac{-8}{25} \quad \text{as taking } s\,' \to -2$$

$$\int_0^\infty t\sin t\, e^{3t}dt = \frac{-6}{50} \quad \text{as taking } s\,' \to -3$$

The last two integrals are incorrect because, $-2, -3$ are not in the domain of convergence of $L\{\sin t\}$.

Problems

Find the Laplace Transform of the following functions

1. t^2e^{-2t}
2. $e^{-at}\sinh bt$
3. $\sinh 3t\cos^2 t$
4. $e^{2t}\cos^2 t$
5. $e^{-3t}(2\cos 5t - 3\sin 5t)$
6. $e^{4t}\sin 2t\cos t$
7. $t\cosh 3t$
8. Show that $\int_0^\infty te^{-2t}\cos t\,dt = \frac{3}{25}$

Topi
cs

1. Laplace Transform of Integrals
2. Multiplication by t^n
3. Division by t
4. Laplace transform of Bessel Function Laplace
5. Transform of Error Function
6. Laplace Transform of Sine and cosine integrals

The Laplace Transform of an Integral

Theorem

If $L\{f(t)\} = F(s)$,
then

$$L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}. \quad (23)$$

Proof.

Let $g(t) = \int_0^t f(\tau)d\tau$ then $g(0) = 0.$ □

Derivatives of the Laplace Transform

Theorem

If $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) \tag{24}$$

where, $n \in \mathbb{N} \cup \{0\}$.

Integral of the Laplace Transform

Theorem

If $L\{f(t)\} = F(s)$,
then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\,ds. \qquad (25)$$

Provided the integral is converges(exists).

Bessel’s function

The linear second order ordinary differential equation of type

$$x^2y'' + xy' + (x^2 - n^2)y = 0, n = 0, 1, 2, \cdots$$

is called the Bessel equation. One of the solution of Bessel’s equation is the function

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!}, |x| < \infty \quad (26)$$

is called the Bessel’s function order n. Bessel’s function of order 0 is given by

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k!(k)!}, |x| < \infty$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots \quad (27)$$

Bessel's function of order 0

Example

Find the Laplace transform of Bessel's function of order 0.

Using the series representation of $J_0(t)$,(27) we obtain

Bessel's function of order 0

Find the Laplace transform of Bessel's function of order 0.

Using the series representation of $J_0(t)$,(27) we obtain

$$L\{J_0(t)\} = L\left\{1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots\right\}$$

$$= L\{1\} - L\left\{\frac{t^2}{2^2}\right\} + L\left\{\frac{t^4}{2^2.4^2}\right\} - L\left\{\frac{t^6}{2^2.4^2.6^2}\right\} + \cdots$$

Bessel's function of order 0

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$$= \frac{1}{s} - \frac{2!}{2^2 s^3} + \frac{4!}{2^2.4^2 s^5} - \frac{6!}{2^2.4^2.6^2 s^7} + \cdots$$

Bessel's function of order 0

Example

Find the Laplace transform of Bessel's function of order 0.

Using the series representation of $J_0(t)$,(27) we obtain

$$L\{J_0(t)\} = L\left\{1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots\right\}$$

$$= L\{1\} - L\left\{\frac{t^2}{2^2}\right\} + L\left\{\frac{t^4}{2^2.4^2}\right\} - L\left\{\frac{t^6}{2^2.4^2.6^2}\right\} + \cdots$$

$$= \frac{1}{s} - \frac{2!}{2^2 s^3} + \frac{4!}{2^2.4^2 s^5} - \frac{6!}{2^2.4^2.6^2 s^7} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^2}\right)^3 + \cdots\right]$$

#

Bessel's function of order 0

Example

Find the Laplace transform of Bessel's function of order 0.

Using the series representation of $J_0(t)$, (27) we obtain

$$
\begin{aligned}
L\{J_0(t)\} &= L\left\{1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots\right\} \\
&= L\{1\} - L\left\{\frac{t^2}{2^2}\right\} + L\left\{\frac{t^4}{2^2.4^2}\right\} - L\left\{\frac{t^6}{2^2.4^2.6^2}\right\} + \cdots \\
&= \frac{1}{s} - \frac{2!}{2^2 s^3} + \frac{4!}{2^2.4^2 s^5} - \frac{6!}{2^2.4^2.6^2 s^7} + \cdots \\
&= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^2}\right)^3 + \cdots\right] \\
&= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+s^2}}
\end{aligned}
$$

$$L\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}$$

$$L\{e^{at}J_0(t)\} = \frac{1}{\sqrt{1+(s-a)^2}}$$ First shifting theorem

$$L\{J_0(at)\} = \frac{1}{\sqrt{a^2+s^2}}$$ Change of scale

$$L\{tJ_0(t)\} = \frac{s}{(1+s^2)^{3/2}}$$ Multiplication by t

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$$L\{tJ_0(t)\} = \frac{s}{(1+s^2)^{3/2}}$$ Multiplication by t

$$L\left\{\frac{J_0(t)}{t}\right\} = ??$$

Error function

The error function, erf(x) is defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \qquad (28)$$

In terms of power series, we have

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} dt$$

Which is same as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}$$

By (28) it is easy to see that

$$\text{erf}(0) = 0 \quad \& \quad \text{erf}(\infty) = 1$$

Complementary error function

The complementary error function is defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt \qquad (29)$$

By equations (28) and (29) it is easy to see that

$$\boxed{\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)}$$

Laplace transform of Error function

Find the $L\{\text{erf}(\sqrt{t})\}$

Since

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\,dx$$

$$\text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2}\,dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left(1 - \frac{x^2}{1} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) dt$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{2!\times 5} - \frac{x^7}{3!\times 7} + \cdots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \left(t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\times 2!} - \frac{t^{7/2}}{7\times 3!} + \cdots\right)$$

cont ···

now, taking the Laplace transform

$$L\{\operatorname{erf}(\sqrt{t})\} = \frac{2}{\sqrt{\pi}} L\left\{ t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5 \times 2!} - \frac{t^{7/2}}{7 \times 3!} + \cdots \right\}$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3 \times s^{5/2}} + \frac{\Gamma(7/2)}{2! \times 5 \times s^{7/2}} - \cdots \right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{4s^{5/2}} + \frac{\sqrt{\pi}}{8s^{7/2}} - \cdots \right]$$

$$= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2}\left(\frac{1}{s}\right) + \frac{1.3}{2.4}\left(\frac{1}{s}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{s}\right)^3 + \cdots \right]$$

$$= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s}\right)^{-\frac{1}{2}} = \frac{1}{s\sqrt{1+s}}$$

cont ···

now, taking the Laplace transform

$$L\{\text{erf}(\sqrt{t})\} = \frac{2}{\sqrt{\pi}} L\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\times 2!} - \frac{t^{7/2}}{7\times 3!} + \cdots \right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3\times s^{5/2}} + \frac{\Gamma(7/2)}{2!\times 5\times s^{7/2}} - \cdots \right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{4s^{5/2}} + \frac{\sqrt{\pi}}{8s^{7/2}} - \cdots \right]$$

$$= \frac{1}{s^{3/2}}\left[1 - \frac{1}{2}\left(\frac{1}{s}\right) + \frac{1.3}{2.4}\left(\frac{1}{s}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{s}\right)^3 + \cdots \right]$$

$$= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-\frac{1}{2}} = \frac{1}{s\sqrt{1+s}} \Rightarrow \boxed{L\{\text{erf}(\sqrt{t})\} = \frac{1}{s\sqrt{1+s}}}$$

Sine and cosine integrals

Definition

Sine and Cosine integral functions are defined by respectively

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt \quad \text{and}$$

$$\text{Ci}(x) = \int_x^\infty \frac{\cos t}{t}\,dt$$

Laplace Transform of Sine Integral: Solution 1

Since, $L\{\sin t\} = \dfrac{1}{1+s^2}$ then equation (25) we have

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{1+s^2}\,ds = \tan^{-1}(s)\Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s) = \tan^{-1}\left(\frac{1}{s}\right)$$

and by the equation (23) we obtain

$$L\left\{\int_0^x \frac{\sin t}{t}\,dt\right\} = \frac{1}{s} \times L\left\{\frac{\sin t}{t}\right\}$$

$$= \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$$ which is same as

$$L\{\mathrm{Si}(x)\} = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$$

Laplace Transform of Sine Integral: Solution 2

Let $f(t) = \int_0^t \frac{\sin u}{u} du$. Then $f(0) = 0$ and $f'(t) = \frac{\sin t}{t}$ or $t f'(t) = \sin t$. Taking the Laplace transform we get

$$L\{t f'(t)\} = L\{\sin t\}$$

$$-\frac{d}{ds}\{sF(s) - f(0)\} = \frac{1}{1+s^2}$$

$$\frac{d}{ds}\{sF(s)\} = \frac{-1}{1+s^2}$$

Integrating we obtain $sF(s) = -\tan^{-1}(s) + c$. By the initial value theorem $\lim_{s\to\infty} sF(s) = f(0)$, then $c = \frac{\pi}{2}$. Hence $sF(s) = \frac{\pi}{2} - \tan^{-1}(s) = \tan^{-1}\frac{1}{s}$.

Therefore, $F(s) = \frac{1}{s}\tan^{-1}\frac{1}{s}$. Which is same as

$$L\{\text{Si}(x)\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Laplace Transform of Sine Integral: Solution 1

Let $f(t) = \int_t^\infty \frac{\cos u}{u}\,du$. Then $f(\infty) = 0$ and $f'(t) = \frac{-\cos t}{t}$ or $t f'(t) = -\cos t$.

Taking the Laplace transform we get

$$L\{t f'(t)\} = L\{-\cos t\}$$

$$-\frac{d}{ds}\{sF(s) - f(0)\} = \frac{-s}{1+s^2}$$

$$\frac{d}{ds}\{sF(s)\} = \frac{s}{1+s^2}$$

Integrating we obtain $sF(s) = \frac{1}{2}\log(1+s^2) + c$. By the final-value theorem $\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t) = 0$, then $c = 0$. Hence $sF(s) = \frac{1}{2}\log(1+s^2)$.

Therefore,

$F(s) = \frac{1}{s}\frac{1}{2}\log(1+s^2)$. Which is same as

$$L\{\text{Ci}(x)\} = \frac{1}{2s}\log(1+s^2)$$

Topics

1. Definition of Inverse Laplace Transform
2. Linearity Property
3. First Shifting Theorem
4. Second Shifting Theorem
5. Change of Scale property
6. partial fractions

Definition:

If the Laplace transform of a function $f(t)$ is $F(s)$, i.e. $L\{f(t)\} = F(s)$, then

$f(t)$ is called an inverse Laplace transform of $F(s)$ and we write $L^{-1}\{F(s)\} = f(t)$.

Definition:

If the Laplace transform of a function $f(t)$ is $F(s)$, i.e. $L\{f(t)\} = F(s)$, then

$f(t)$ is called an inverse Laplace transform of $F(s)$ and we write $L^{-1}\{F(s)\} = f(t)$. **Example:**

Since, $L\{1\} = \frac{1}{s}$, we can write $L^{-1}\left\{\frac{1}{s}\right\} = 1$.

$L\{e^{4t}\} = \frac{1}{s-4}$, $s > 4$, we can write $L^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}$.

Table of Inverse Laplace Transforms

$$L\{1\} = \frac{1}{s}$$

$$L\{1\} = \frac{1}{s} \Rightarrow L^{-1} \frac{1}{s} = 1$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

Table of Inverse Laplace Transforms

$$L\{1\} = \frac{1}{s} \Rightarrow L^{-1}\frac{1}{s} = 1$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\frac{n!}{s^{n+1}} = t^n$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

Table of Inverse Laplace Transforms

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$$L\{1\} = \frac{1}{s} \Rightarrow L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

$$L\{e^{at}\} = \frac{1}{s-a} \Rightarrow L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$L\{\cos(bt)\} = \frac{s}{s^2+b^2}$$

Table of Inverse Laplace Transforms

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$$L\{\cos(bt)\} = \frac{s}{s^2+b^2} \Rightarrow L^{-1}\frac{s}{s^2+b^2} = \cos(bt)$$

$$L\{\sin(bt)\} = \frac{b}{s2+b2}$$

Table of Inverse Laplace Transforms

$$L\{1\} = \frac{1}{s} \Rightarrow L^{-1}\frac{1}{s} = 1$$

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$$L\{\cosh(bt)\} = \frac{s}{s2-b2}$$

Table of Inverse Laplace Transforms

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Table of Inverse Laplace Transforms

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$$L\{\cos(bt)\} = \frac{s}{s^2+b^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos(bt)$$

$$L\{\sin(bt)\} = \frac{b}{s^2+b^2} \Rightarrow L^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin(bt)$$

$$L\{\cosh(bt)\} = \frac{s}{s^2-b^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2-b^2}\right\} = \cosh(bt)$$

$$L\{\sinh(bt)\} = \frac{b}{s^2-b^2} \Rightarrow L^{-1}\left\{\frac{b}{s^2-b^2}\right\} = \sinh(bt)$$

Linearity property

Let $F_1(s)$ and $F_2(s)$ be the Laplace transform of $f_1(t)$ and $f_2(t)$ respectively and $c_1, c_2 \in \mathbb{R}$, then

$$\begin{aligned} L^{-1}\{c_1F_1(s) + c_2F_2(s)\} &= c_1L^{-1}\{F_1(s)\} + c_2L^{-1}\{F_2(s)\} \\ &= c_1f_1(t) + c_2f_2(t) \end{aligned}$$

First shifting theorem

[redacted]

If $L^{-1}\{F(s)\} = f(t)$, *then*

$$L^{-1}\{F(s-a)\} = e^{at} f(t)$$

Second shifting theorem

Theorem

If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\{e^{-as}F(s)\} = \begin{cases} f(t-a), & t > a; \\ 0, & t < a. \end{cases}$$

Change of scale property

Theorem

If $L^{-1}\{F(s)\} = f(t)$, *then*

$$L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right).$$

Partial Fractions

Partial Fraction Decompositions. We will be concerned with the quotient of two polynomials, namely a rational function

$$F(s) = \frac{P(s)}{Q(s)}$$

where the degree of $Q(s)$ is greater than the degree of $P(s)$, and $P(s)$ and $Q(s)$ have no common factors. Then $F(s)$ can be expressed as a finite sum of partial fractions.

- For each linear factor of the form $as + b$ of $Q(s)$, there corresponds a partial fraction of the form

$$\frac{A}{as+b}, \quad A \text{ is a constant}$$

Cont ...

1. For each repeated linear factor of the form $(as+b)^n$, there corresponds a partial fraction of the form

$$\frac{A_1}{as+b}+\frac{A_2}{(as+b)^2}+\cdots+\frac{A_n}{(as+b)^n}, \quad A_1, A_2, \cdots A_n \quad \text{constants}$$

Cont ...

1. For each repeated linear factor of the form $(as+b)^n$, there corresponds a partial fraction of the form

$$\frac{A_1}{as+b}+\frac{A_2}{(as+b)^2}+\cdots+\frac{A_n}{(as+b)^n}, \quad A_1, A_2, \cdots A_n \text{ constants}$$

2. For every quadratic factor of the form as^2+bs+c, there corresponds a partial fraction of the form

$$\frac{As+B}{as^2+bs+c}, \quad A, B \text{ constants}$$

Cont ...

- For every repeated quadratic factor of the form $(as^2 + bs + c)^n$, there corresponds a partial fraction of the form

$$\frac{A_1 s + B_1}{as^2 + bs + c} + \frac{A_2 s + B_2}{(as^2 + bs + c)^2} + \cdots + \frac{A_n s + B_n}{(as^2 + bs + c)^n}$$

$A_1, A_2, \cdots A_n, B_1, B_2, \ldots B_n$ constants

Definition and Examples

Find: $L^{-1}\left[\frac{4s+1}{s^2+10s+34}\right]$

Sol: Here the denominator does not factor over the reals. Hence complete the square.

$$s^2+10s+34 = s^2+10s+25-25+34 = (s+5)^2+9$$

$$L^{-1}\left[\frac{4s+1}{s^2+10s+34}\right] = L^{-1}\left[\frac{4s+1}{(s+5)^2+9}\right] = L^{-1}\left[\frac{4(s+5)-20+1}{(s+5)^2+9}\right]$$

$$= L^{-1}\left[\frac{4(s+5)-19}{(s+5)^2+9}\right]$$

$$= 4L^{-1}\left[\frac{(s+5)}{(s+5)^2+9}\right] - 19L^{-1}\left[\frac{1}{(s+5)^2+9}\right]$$

$$= 4e^{-5t}\cos(3t) - \frac{19}{3}L^{-1}\left[\frac{3}{(s+5)^2+9}\right]$$

$$= 4e^{-5t}\cos(3t) - \frac{19}{3}e^{-5t}\sin(3t)$$

Topi
cs

1. Inverse Laplace transforms of Derivatives Laplace
2. Transforms of Integrals
3. by Powers of P- Division by powers of P Convolution
4. Theorem
5. Heaviside's Expansion theorem

Inverse Laplace Transform of derivatives

If $L^{-1}\{F(s)\} = f(t)$,
then

$$L^{-1}\{F^{(n)}(s)\} = L^{-1}\left\{\frac{d^n}{ds^n}F(s)\right\} = (-1)^n t^n f(t)$$

Inverse Laplace Transform of Integrals

If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\left\{\int_s^{\infty} F(u)du\right\} = \frac{f(t)}{t}$$

Inverse Laplace Transform of derivatives

Inverse Laplace Transform of derivatives

Examples

Show that $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$.

Solution:

Let $g(t) = \int_0^t f(\tau)d\tau$. Then $g'(t) = f(t)$ and $g(0) = 0$. By Laplace transform of derivatives

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$
$$L\{f(t)\} = sL\{g(t)\}$$
$$F(s) = sL\{g(t)\}$$
$$\frac{F(s)}{s} = L\{g(t)\}$$

which is same as

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

Examples

Show that $L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \int_0^{t_1} f(\tau)\,d\tau\,dt_1$.

Solution:

Let $g(t) = \int_0^t \int_0^{t_1} f(\tau)\,d\tau\,dt_1$. Then $g'(t) = \int_0^t f(\tau)\,d\tau$ and $g''(t) = f(t)$. Since $g(0) = g'(0) = 0$. By Laplace transform of derivatives

$$L\{g''(t)\} = s^2 L\{g(t)\} - sg(0) - g'(0)$$

$$L\{f(t)\} = s^2 L\{g(t)\}, \quad \text{since } g''(t) = f(t)$$

$$F(s) = s^2 L\{g(t)\}$$

$$\frac{F(s)}{s^2} = L\{g(t)\} \quad \text{same as}$$

$$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \int_0^{t_1} f(\tau)\,d\tau\,dt_1$$

Convolution:

Application 1: Calculating Integrals

Show that $\int_0^\infty te^{-3t}\sin t\,dt = \frac{3}{50}$

Since, $L\{\sin t\} = \frac{1}{1+s^2}$ and

$$L\{t\sin t\} = -\frac{d}{ds}L\{\sin t\} = -\frac{d}{ds}\frac{1}{1+s^2} = \frac{2s}{(1+s^2)^2}$$

equivalently,

$$\int_0^\infty t\sin t e^{-st}dt = \frac{2s}{(1+s^2)^2}$$

Now taking the limit as $s \to 3$

$$\int_0^\infty t\sin t e^{-3t}dt = \frac{2\times 3}{(1+3^2)^2} = \frac{3}{50}$$

Application 1: Calculating Integrals

since, $L\{t\sin t\} = \int_0^\infty t\sin t\, e^{-st}dt = \frac{2s}{(1+s^2)^2}$

1. $\int_0^\infty t\sin t\, e^{-3t}dt = \frac{3}{50}$
2. $\int_0^\infty t\sin t\, e^{-t}dt = \frac{1}{2}$ for $s = 1$
3. $\int_0^\infty t\sin t\, dt = 0$? for $s = 0$
4. $\int_0^\infty t\sin t\, e^{3t}dt = \frac{-3}{50}$? $s = -3$

Application 2: Solving Differential Equations

Solve the initial value problem $y'' + 4y = \cos(2t), y(0) = 1, y'(0) = 1$. Solution.

Taking the Laplace of both sides to obtain $L\{y''\} + 4L\{y\} = L\{\cos(2t)\}$ the last equation reduces to

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{s}{s^2+4}$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{s+1}{s^2+4} + \frac{s}{(s^2+4)} = \frac{s}{s^2+4} + \frac{1}{s^2+4} + \frac{s}{(s^2+4)^2}$$

taking Laplace inverse we get

$$y(t) = L^{-1}\frac{s}{s^2+4} + L^{-1}\frac{1}{s^2+4} + L^{-1}\frac{s}{(s^2+4)^2}$$

$$= \cos(2t) + \frac{\sin(2t)}{2} + \frac{t}{4}\sin(2t)$$

Application 3: Solving Partial Differential Equations

Application 4: Integral Equations

Application 5: Circuite Problems

Application: Calculating Integrals

Convergence of Integrals

Integrals with Infinite limits of integration are called **improper integral of Type-I**

If $f(x)$ is continuous on $[a, \infty)$,

$$\int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

If the limit is finite we ay that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exists, the improper integral **diverges**.

Example-I

Is the integral $\int_0^\infty \frac{1}{1+x^2}dx$ converges?

$$\int_0^\infty \frac{1}{1+x^2}dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}dx$$

$$= \lim_{b\to\infty} \tan^{-1}(x)\Big|_0^b$$

$$= \lim_{b\to\infty} \tan^{-1}(b) - \tan^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

So, limit value is finite, hence the above integral is converges.

Example-II

Is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges?

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx$$
$$= \lim_{b\to\infty}\left[\frac{1}{(1-p)x^{1-p}}\right]_0^b$$
$$= \lim_{b\to\infty}\frac{1}{(1-p)b^{P-1}} - \frac{1}{1-p} = \begin{cases}\frac{1}{p-1}, & p>1;\\ \infty, & p<1.\end{cases}$$

So, limit value is finite, hence the above integral is converges.

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases}\text{converges}, & \text{if } p>1;\\ \text{diverges}, & \text{if } p\le 1.\end{cases}$$

Direct Comparison Test:

Theorem

Let f and g are continuous function on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_0^\infty f(x)dx$ converges if $\int_0^\infty g(x)dx$ converges.
2. $\int_0^\infty g(x)dx$ diverges if $\int_0^\infty f(x)dx$ converges.

Order 0 and 1

- When $n = 0$ in (4), Bessel's function of order 0 is given by

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \cdots \quad (30)$$

- When $n = 1$ in (4), Bessel's function of order 1 is given by

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 . 3} + \frac{x^5}{2^2 . 4^2 . 6} - \cdots \quad (31)$$

Limit Comparison Test:

Theorem

If the positive functions f and g are continuous on $[a,\infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

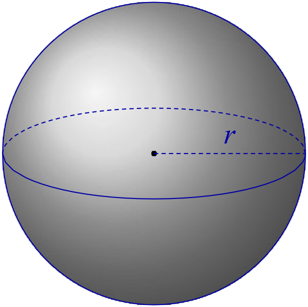
then $\int_0^{\infty} f(x)\,dx$ *and* $\int_0^{\infty} g(x)\,dx$

both converge or both diverge.

References

L. Debnath, D. Bhatta, **Integral Tranforms and their Applications**, Vol. 3. CRC Press, New York 2015.

1. $k\,dx = kx + c$
2. $\frac{1}{x}dx = \ln x + c$
3. $x^n dx = \frac{x^{n+1}}{n+1} + c, (n \neq -1)$
4. $e^x dx = e^x + c$
5. $a^x dx = \frac{a^x}{\ln a} + c$
6. $\cos x dx = \sin x + c$
7. $\sin x dx = -\cos x + c$
8. $\sec^2 x dx = \tan x + c$
9. $\csc^2 x dx = -\cot x + c$
10. $\sec x \tan x dx = \sec x + c$
11. $\csc x \cot x dx = -\csc x + c$
12. $\tan x dx = \ln|\sec x| + c$
13. $\cot x dx = \ln|\sin x| + c$
14. $\sinh x dx = \cosh x + c$
15. $\cosh x dx = \sinh x + c$
16. $\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$
17. $\frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + c$
18. $\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} +$
19. $\frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + c$
20. $\frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + c$
21. $\ln x dx = x\ln x - x + c$
22. $e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c$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Thank You !

UNIT-I : Curve Fitting:

Byvariate data:- In statistics, byvariate data is an data on each of two variables, where each values of one of the variable is dependent, another one is independent variable.

Main objective of byvariate data is to estimate relationship b/w two variables.

(1) **Curve Fitting:-** Curve Fitting to fit a equation and draw the curves (linear ~~& non-linear~~) is called as "Curve Fitting"

Statistical methods like Measure of central tendency, dispersion, Skewness, Kurtosis etc are used in the study of single variable. In most of the business and economic phenomena it is required to deal with situation involving more than one variable. It is observe that there exists a relationship between two or more variable in this phenomena.

For ex:- Solving and investment, price & demand, price & supply etc..

In curve fitting we write to establish a mathematical relationship between can be expressed in the form of functional relationship b/w the related variable. The variabl various reation are

C F classified in two types

① Linear Form

② Non-Linear form

4 mar Imp

Principles of least squares (or) Legender's:-

Principles :- Let $y = f(x)$ be the functional relationship b/w the two variables, x and y. where x is independent variable and y is depedent variable. For any of value of x, the value of y can be estimated using the relationship $y = f(x)$.

Let the estimated value of y is denoted as $\hat{y}$. The difference between the exact value of y and estimated value of $\hat{y}$ is known as "Residue" that is $y - \hat{y}$. The residue sum of square is denoted by "S" and is given by

$S = \Sigma (y - \hat{y})^2$.

The principles of least squares states that the residue sum of square should be minim... That is the S should be value of s is minimum, wher... the d/f b/w exact value of y and estima... value is near to exact value.

∴ Any relationship b/w the variable x & y with th...

help of least square provided best estimate possible.

Fitting of Straight line of the Form $y = ax+b$

Let $y = ax+b$ be the straight line eq where a, b Constants, which are to be estimated

$$\hat{y} = ax+b$$

principles of least squares states that reside sum of squares should be minimum "S" is minimum.

$$S = \Sigma (y-\hat{y})^2$$

$$S = \Sigma (y-(ax+b))^2$$

$$S = \Sigma (y-ax-b)^2$$

Using minimum conditions

$$\frac{\partial s}{\partial a} = 0, \frac{\partial^2 s}{\partial a^2} > 0$$

$$\frac{\partial s}{\partial b} = 0, \frac{\partial^2 s}{\partial b^2} > 0$$

$$\frac{\partial s}{\partial a} = 0 \Rightarrow \frac{\partial}{\partial a} \Sigma (y-ax-b)^2 = 0$$

$\because \frac{d}{dx}(x^2) = 2x$

$$\Rightarrow 2 \Sigma (y-ax-b)(-x) = 0$$

(ax)

$$\Rightarrow -2 \Sigma xy + 2a \Sigma x^2 + 2b \Sigma x = 0$$

$\frac{d}{da}(-ax)$

$$\Rightarrow 2(-\Sigma xy + a \Sigma x^2 + b \Sigma x) = 0$$

$(1)(-x)$

$$\Rightarrow -\Sigma xy + a \Sigma x^2 + b \Sigma x = 0$$

$$\Rightarrow \Sigma xy = a \Sigma x^2 + b \Sigma x \quad \text{--- (1)}$$

$$\frac{\partial^2 s}{\partial a^2} > 0 = \frac{\partial}{\partial a}\left(\frac{\partial s}{\partial a}\right) = \frac{\partial}{\partial a}\left(-2\Sigma_i xy + 2a\Sigma_i x^2 + 2b\Sigma_i x\right)$$

$$\frac{\partial^2 s}{\partial a^2} = 2\Sigma_i x^2 > 0$$

$$\frac{\partial^2 s}{\partial a^2} > 0$$

$$\frac{\partial s}{\partial b} = 0 \Rightarrow \frac{\partial}{\partial b}\Sigma_i (y - ax - b)^2 = 0$$

$$\Rightarrow 2\Sigma_i (y - ax - b)(-1) = 0$$

$$\Rightarrow -2\Sigma_i y + 2a\Sigma_i x + 2\Sigma_i b = 0 \; [\because \Sigma_i = n]$$

$$\Rightarrow 2(-\Sigma_i y + a\Sigma_i x + nb) = 0$$

$$\Rightarrow -\Sigma_i y + a\Sigma_i x + nb = 0 \quad (2)$$

$$\Sigma_i y = a\Sigma_i x + nb \quad (2)$$

$$\frac{\partial^2 s}{\partial s^2} > 0 \Rightarrow \frac{\partial}{\partial b}\left(\frac{\partial s}{\partial b}\right) = \frac{\partial}{\partial b}\left(-2\Sigma_i y + 2a\Sigma_i x + 2nb\right)$$

$$= \frac{\partial}{\partial b}(2nb) \qquad \because \frac{\partial s}{\partial b} = 1$$

$$= 2n > 0$$

The above equation

∴ (1) & (2) are called normal equations

∴ The required fitted equation is $\hat{y} = \hat{a}x + \hat{b}$

Fitting of Stright line of the form $y = a + bx$ — (1)

1: Let $y = a + bx$ be the Stright line equation where a, Constants, which are to be estimated

$$\hat{y} = a + bx$$

Principles of least square states that residue S of square should be minimum "s" is minimum

$S = \sum (y - \hat{y})^2$

$S = \sum (y - (a+bx))^2$

$S = \sum (y - a - bx)^2$

using minimum conditions

$\frac{\partial s}{\partial a} = 0, \frac{\partial^2 s}{\partial a^2} > 0$

$\frac{\partial s}{\partial b} = 0, \frac{\partial^2 s}{\partial s^2} > 0$

$\because \frac{d}{dn}(x^2) = 2x(1)$

$\frac{\partial s}{\partial a} = 0 \Rightarrow \frac{\partial}{\partial a} \sum (y - a - bx)^2 = 0$

$\Rightarrow 2\sum (y - a - bx)(-1) = 0$

$\Rightarrow -2\sum y + 2a\sum + 2b\sum x = 0$ $(\sum = n)$

$\Rightarrow 2(-\sum y + na - b\sum x) = 0$

$\Rightarrow -\sum y + na + b\sum x = 0$

$\Rightarrow \sum y = na + b\sum x$ —— ①

$\frac{\partial^2 s}{\partial a^2} > 0 \Rightarrow \frac{\partial}{\partial a}\left(\frac{\partial s}{\partial a}\right) \Rightarrow \frac{\partial}{\partial a}(-2\sum y + 2na + 2b\sum x)$

$\Rightarrow 2n > 0$

$\frac{\partial^2 s}{\partial a^2} > 0$

$\frac{\partial s}{\partial b} = 0 \Rightarrow \frac{\partial}{\partial b} \sum (y - a - bx)^2$

$\Rightarrow 2\sum (y - a - bx)(-x) = 0$

$\Rightarrow -2\sum xy + 2a\sum x + 2b\sum x^2 = 0$

$\Rightarrow 2(-\sum xy + a\sum x + b\sum x^2) = 0$

$\Rightarrow -\sum xy + a\sum x + b\sum x^2 = 0$

$\Rightarrow \sum xy = a\sum x + b\sum x^2$ —— ②

$\frac{\partial^2 s}{\partial s^2} > 0$

$$\frac{\partial}{\partial s}\left(\frac{\partial s}{\partial b}\right) = \frac{\partial}{\partial b}\left(-2\sum x y + 2a\sum x + 2b\sum x^2\right)$$

$$= 2\sum x^2 > 0$$

$$\frac{\partial^2 s}{\partial b^2} > 0$$

The above equation ① & ② are called normal equations

∴ The required fitted equation is $\hat{y} = \hat{a} + \hat{b}x$

① Fit a straight line for the following data $(y = a + bx)$

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

Sol:- let straight line equation is $y = a + bx$

x	y	x^2	xy
1	1	1	1
3	2	9	6
4	4	16	16
6	4	36	24
8	5	64	40
9	7	81	63
11	8	121	88
14	9	196	126
56	40	524	364

$n = 8$

$\sum y = na + b\sum x$ — ①

$\sum xy = a\sum x + b\sum x^2$ — ②

$40 = 8a + 56b$ — ③ ×7

$364 = 56a + 524b$ — ④

$280 = 56a + 392b$

$364 = 56a + 524b$

(−) (−) (−)

$-84 = -132b$

$84 = 132b$

$b = \frac{84}{132}$

$\hat{b} = 0.6363$

Sub $\hat{b}$ in equation ③

$40 = 8a + 56(0.6363)$

$40 = 8a + 35.6328$

$40 = 8a + 35.6328$

$40 - 35.6328 = 8a$

$\frac{4.3672}{8} = a$ $\boxed{\hat{a} = 0.5459}$

Q. Fit a straight line of the form $y = a + bx$ to the following data

x	2	4	6	8	10	12
y	10	14	19	25	31	36

x	y	x^2	xy
2	10	4	20
4	14	16	56
6	19	36	114
8	25	64	200
10	31	100	310
12	36	144	432
42	135	364	1132

$n = 6$

$\Sigma y = na + b\Sigma x$ —(1)

$\Sigma xy = a\Sigma x + b\Sigma x^2$ —(2)

$135 = 6a + 42b$ —(3) ×7

$1132 = 42a + 364b$ —(4)

$945 = 42a + 294b$

$1132 = 42a + 364b$

$-187 = -70$

$b = \frac{187}{70}$

$\boxed{\hat{b} = 2.6714}$

Sub $\hat{b}$ in eq (3)

$135 = 6a + 42(2.6714)$

$135 = 6a + 112.1988$

$135 - 112.1988 = 6a$

$22.8012 = 6a$

$\bar{a} = \frac{22.8012}{6}$ $\boxed{\hat{a} = 3.8002}$

Fitting of second degree parabola polynomial of the form $y = ax^2+bx+c$

Let $y = ax^2+bx+c$ be the equation of a second degree parabola where a, b, c constants. which are to be estimated. Let the pair of observations are considered into a linear form. Let (x_1, y_1), (x_2, y_2) so on. (x_n, y_n) be the pair of observations observed on two variables x and y.

The principle of least squares states that the residue sum of square should be minimum i.e

$S = \Sigma(y-\hat{y})^2$ s

Let estimated value of y is denoted

as $\hat{y} = ax^2+bx+c$

$$\Sigma(y-\hat{y})^2 = \Sigma(y-ax^2-bx-c)^2$$

using minimum conditions

$$\frac{\partial s}{\partial a} = 0, \quad \frac{\partial^2 s}{\partial a^2} > 0$$

$$\frac{\partial s}{\partial b} = 0, \quad \frac{\partial^2 s}{\partial b^2} > 0, \quad \frac{\partial s}{\partial c} = 0, \quad \frac{\partial^2 s}{\partial c^2} > 0$$

$$\frac{\partial s}{\partial a} = 0 \Rightarrow \frac{\partial}{\partial a}\Sigma(y-ax^2-bx-c)^2 = 0$$

$$2(y-ax^2-bx-c)(-x^2) = 0$$

$$-2\Sigma yx^2 + 2\Sigma ax^4 + 2\Sigma bx^3 + 2\Sigma cx^2 = 0$$

$$\Sigma x^2 y = \Sigma a x^4 + \Sigma b x^3 + \Sigma c x^2 \longrightarrow (1)$$

$$\frac{\partial^2 s}{\partial a^2} > 0 \Rightarrow \frac{\partial}{\partial a}\left(-2\Sigma y x^2 + 2\Sigma a x^4 + 2\Sigma b x^3 + 2\Sigma c x^2\right) > 0$$

$$2\Sigma x^4 > 0 \Rightarrow \frac{\partial^2 s}{\partial b^2} > 0$$

$$\frac{\partial s}{\partial b} = 0 \Rightarrow \frac{\partial}{\partial b}(y - ax^2 - bx - c)^2 = 0$$

$$2\Sigma(y - ax^2 - bx - c)(-x) = 0$$

$$-2\Sigma xy + 2a\Sigma x^3 + 2b\Sigma x^2 + 2c\Sigma x = 0$$

$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x \longrightarrow (2)$$

$$\frac{\partial^2 s}{\partial b^2} = \frac{\partial}{\partial b}\left[-2\Sigma xy + 2a\Sigma x^3 + 2b\Sigma x^2 + 2c\Sigma x\right] > 0$$

$$2\Sigma x^2 > 0 \Rightarrow \frac{\partial^2 s}{\partial b^2} > 0$$

$$\frac{\partial s}{\partial c} = 0 \Rightarrow \frac{\partial}{\partial c}\Sigma(y - ax^2 - bx - c)^2 = 0$$

$$2\Sigma(y - ax^2 - bx - c)(-1) = 0$$

$$-2\Sigma y + 2a\Sigma x^2 + 2b\Sigma x + 2nc = 0$$

$$\Sigma y = a\Sigma x^2 + b\Sigma x + nc \longrightarrow (3)$$

$$\frac{\partial^2 s}{\partial c^2} > 0 \Rightarrow \frac{\partial}{\partial c}\left(-2\Sigma y + 2a\Sigma x^2 + 2b\Sigma x + 2nc\right) = 0$$

$$2n > 0 \Rightarrow \frac{\partial s^2}{\partial c^2} > 0$$

The above (1), (2) and (3) are known as normal eq^{n}s

Solving three normal eq^{n}s we get $\hat{a}; \hat{b}; \hat{c}$

∴ The required fitting eq^n is $y = \hat{a}x^2 + \hat{b}x + \hat{c}$

Fitting of second degree parabola or polynomial of the Form $y = a + bx + cx^2$

Let $y = a + bx + cx^2$ be the eq^n of second degree parabola. where a,b,c are constants which are to be estimated. Let the pair of observations are considered into a linear form. Let $(x_1, y_1)(x_2, y_2) \dots (x_n, y_n)$ be the pair of observations observed on two variables x and y.

The principal of least squares states that "the residence sum of squares should be minimum" i.e,

$S = \Sigma(y - \hat{y})^2$ should be minimum

Let the estimated value $\hat{y}$ is denoted as $\hat{y} = a + bx$

$$S = \Sigma(y - \hat{y})^2 = \Sigma(y - a - bx - cx^2)^2$$

By using minimum conditions $\frac{\partial S}{\partial a} = 0$; $\frac{\partial^2 S}{\partial a^2} > 0$

$\frac{\partial S}{\partial b} = 0$; $\frac{\partial^2 S}{\partial b^2} > 0$

$\frac{\partial S}{\partial c} = 0$; $\frac{\partial^2 S}{\partial c^2} > 0$

$$\frac{\partial S}{\partial a} = 0 \Rightarrow \frac{\partial}{\partial a} \Sigma(y - a - bx - cx^2)^2 = 0$$

$$2\Sigma(y - a - bx - cx^2)(-1) = 0$$

$$-2\Sigma y + 2an + 2b\Sigma x + 2c\Sigma x^2 = 0$$

$$\Sigma y = an + b\Sigma x + c\Sigma x^2 \rightarrow (1)$$

$$\frac{\partial^2 S}{\partial a^2} \Rightarrow \frac{\partial}{\partial a} \Sigma(y - a - bx - cx^2)^2$$

$$\frac{\partial}{\partial a}(-2\Sigma y + 2an + 2b\Sigma x + 2c\Sigma x^2)$$

$$2n > 0 \Rightarrow \frac{\partial^2 S}{\partial a^2} > 0$$

$$\frac{\partial s}{\partial b} = 0 \Rightarrow \frac{\partial}{\partial b} \Sigma_i (y-a-bx-cx^2)^2 = 0$$

$$2\Sigma_i (y-a-bx-cx^2)(-x) = 0$$

$$-2\Sigma_i xy + 2a\Sigma_i x + 2b\Sigma_i x^2 + 2c\Sigma_i x^3 = 0$$

$$\Sigma_i xy = a\Sigma_i x + b\Sigma_i x^2 + c\Sigma_i x^3 \rightarrow (2)$$

$$\frac{\partial s^2}{\partial b^2} \Rightarrow \frac{\partial}{\partial b}(-2\Sigma_i xy + 2a\Sigma_i x + 2b\Sigma_i x^2 + 2c\Sigma_i x^3)$$

$$\frac{\partial^2 s}{\partial b^2} = 2\Sigma_i x^2 > 0$$

$$\frac{\partial^2 s}{\partial b^2} > 0$$

$$\frac{\partial s}{\partial c} = \frac{\partial}{\partial c} \Sigma_i (y-a-bx-cx^2)^2 = 0$$

$$2\Sigma_i (y-a-bx-cx^2)(-x^2) = 0$$

$$-2\Sigma_i x^2 y + 2a\Sigma_i x^2 + 2b\Sigma_i x^3 + 2c\Sigma_i x^4 = 0$$

$$\Sigma_i x^2 y = a\Sigma_i x^2 + b\Sigma_i x^3 + c\Sigma_i x^4 \rightarrow (3)$$

$$\frac{\partial^2 s}{\partial c^2} > 0 \Rightarrow \frac{\partial}{\partial c}(-2\Sigma_i x^2 y + 2a\Sigma_i x^2 + 2b\Sigma_i x^3 + 2c\Sigma_i x^4) >$$

$$2\Sigma_i x^4 > 0$$

$$\frac{\partial^2 s}{\partial c^2} > 0$$

From (1), (2) & (3) are known as normal eqns

Solving three normal eqns we get $\hat{a}, \hat{b}, \hat{c}$

The required fitted eqn is $y = \hat{a} + \hat{b}x + \hat{c}x^2$

① Fit a second degree parabola ... for the following data

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Sol:- Let the second degree parabola is $y = a + bx + cx^2$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
10	12.9	30	100	354	37.1	130.3

$\sum y = na + b\sum x + c\sum x^2$ ——— ①

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$ ——— ②

$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4$ ——— ③

$12.9 = 5a + 10b + 30c$ ——— ④ $\times 2$

$37.1 = 10a + 30b + 100c$ ——— ⑤ $\times 3$

$130.3 = 30a + 100b + 354c$ ——— ⑥

Solving ④ & ⑤

$25.8 = 10a + 20b + 60c$

$37.1 = 10a + 30b + 100c$

(–) (–) (–)

$-11.3 = -10b - 40c$ ——— ⑦

Solving ⑤ & ⑥

$111.3 = 30a + 90b + 300c$

$130.3 = 30a + 100b + 354c$

(–) (–) (–) (–)

$-19 = -10b - 54c$ ——— ⑧

Solving ⑦ & ⑧

$-11.3 = -10b - 40c$

$-19.0 = -10b - 54c$

(+) (+) (+)

$7.7 = 14c$

$c = \frac{7.7}{14}$

$\boxed{\hat{c} = 0.55}$

Sub $\hat{c}$ in eqn ⑦

$-11.3 = -10b - 40(0.55)$

$-11.3 = -10b - 22$

$-11.3 + 22 = -10b$

$10.7 = -b$

$-b = 1.07$

$\boxed{\hat{b} = -1.07}$

Sub $\hat{b}, \hat{c}$ in eq ④

$12.9 = 5a + 10(-1.07) + 30(0.55)$

$12.9 = 5a - 10.7 + 16.5$

$12.9 = 5a + 5.8$

$12.9 - 5.8 = 5a$

$7.1 = 5a$

$a = \frac{7.1}{5}$

$\boxed{\hat{a} = 1.42}$

∴ The required fitting eqn $y = \hat{a} + \hat{b}x + \hat{c}x^2$

$y = 1.42 + (-1.07)x + (0.55)x^2$

$\log a = A$

$\log x = x$

$y = A + bx$ which is the eqⁿ of the straight line, the constants A, B are estimated from the normal eqⁿs with the help of principal least squares

$$\sum_i y = nA + b\sum_i x \quad \text{--- (1)}$$

$$\sum_i xy = A\sum_i x + b\sum_i x^2 \quad \text{--- (2)}$$

Solving the normal eqⁿs we get $\hat{A}$, $\hat{b}$

$\therefore \log a = A \Rightarrow \hat{a} = \text{anti}\log(\hat{A})$

$\therefore$ The required fitting eqⁿ is $y = \hat{a}x^{\hat{b}}$

(5) Fitting of exponential curve of the form $y = ab^x$

Let, the eqⁿ of the exponential curve $y = ab^x$ where a, b are constants which are to be estimated. We transform the above eqⁿ into a straight line form.

Taking 'log' on both sides

$$\log y = \log a + x\log b$$

Put $\log y = y$,

$\log a = A$

$\log b = B$

$y = A + Bx$

which is the eqⁿ of straight line. The constants A, B are estimated from the following normal eqⁿs with the help of principle of least square

$$\sum_i y = nA + B\sum_i x \quad \text{--- (1)}$$

$$\sum_i xy = A\sum_i x + B\sum_i x^2 \quad \text{--- (2)}$$

Solving these two normal eqⁿs we get $\hat{A}$, $\hat{B}$

$$\log a = \hat{A} \Rightarrow a = \text{antilog}(\hat{A})$$

$\log b = B^{\wedge} \Rightarrow \hat{b} = \text{antilog}(B^{\wedge})$ $\hat{a}\hat{b}^{x}$

$\therefore$ The required fitting eq^n is $y = a^{\wedge} b^{\wedge x}$

(c) Fitting of exponential curve

$y = ae^{bx}$

Let, the eq^n of the exponential curve $y = ae^{bx}$ where a, b are constants which to be estimated. we transform the above eq^n into a straight line form

Taking 'log' on both sides

Put $\log y = y$ $\rightarrow$ $\log y = \log a + x \log_e b$

$\log a = A$

$b \log e = B$ $\quad y = A + Bx$

which is the required straight line

$\therefore$ The constants A, B are estimated from the following normal eqs with the help of pri of least squares.

$\therefore$ The normal eqs are $\Sigma y = nA + B\Sigma x \longrightarrow (1)$

$\Sigma xy = A\Sigma x + B\Sigma x^2 \longrightarrow (2)$

Sloving these two normals eq's we get $\hat{A}\,\hat{a}$

$\log \hat{a} = \hat{A} \Rightarrow \hat{a} = A.\alpha[\hat{A}]$

$B \log e = \hat{B} \Rightarrow \hat{b} = \frac{\hat{B}}{\log e} = \frac{\hat{B}}{\log(2.782)} = \frac{\hat{B}}{0.4345}$

$\therefore$ The required fitted eq^n is $\hat{y} = \hat{a}e^{\hat{b}x}$

Date 23/07/24

(1) Fit a power curve of the form $y = ax^b$ to the following data.

x	1	2	3	4	5	6
y	1200	900	600	200	110	50

x	y	X = log y	x^2	xy
1	1200	3.0791	0	0
2	900			

wrong

x	y	X = log x	Y = log y	X^2	XY
1	1200	0	3.0791	0	0
2	900	0.3010	2.9542	0.0906	0.8892
3	600	0.4771	2.7781	0.2276	1.3254
4	200	0.6020	2.3010	0.3624	1.3852
5	110	0.6989	2.0413	0.4884	1.4266
6	50	0.7781	1.6989	0.6054	1.3219
21	3060	2.8571	14.8526	1.7744	6.3483

The power curve eqn is $y = ax^b$

The normal eqns are $\Sigma Y = nA + b\Sigma X \rightarrow (1)$

$\Sigma XY = A\Sigma X + b\Sigma X^2 \rightarrow (2)$

eq-(1)

$14.8526 = 6A + 2.8571b \rightarrow (3) \times 2.8571$

eq-(2)

$6.3483 = 2.8571A + 1.7744b \rightarrow (4) \times 6$

Solving eqn (3) & (4)

(3) × 2.8571, (4) × 6

$42.4353 = 17.1426A + 8.1630b$

$38.0898 = 17.1426A + 10.6464b$

$4.3455 = -2.4834b$

$b = \frac{-4.3455}{2.4834}$

$\hat{b} = -1.7498$

Substitute $\hat{b}$ in eq^n (3)

$14.8526 = 6A + 2.8571(-1.7498)$

$14.8526 = 6A - 4.9993$

$14.8526 + 4.9993 = 6A$

$6A = 19.8519 \Rightarrow A = \frac{19.8519}{6}$

$\boxed{\hat{A} = 3.3086}$

$\hat{a} = A.L(\hat{A})$

$= A.L(3.3086) \longrightarrow$ shift + log = Anti log value

$\boxed{\hat{a} = 2035.1667}$

Required fitted eq^n is $y = \hat{a}x^{\hat{b}}$

$y = (2035.1667)x^{(-1.7498)}$

H.W. fit a power curve of the form $y = ax^b$ to the following data.

x	1	2	3	4	5
y	2	16	54	128	250

Ans:- $\hat{a} = 1.999$

$\hat{b} = 3.003$.

x	y	logx = X	logy = Y	X^2	XY
1	2	0	0.3010	0	0
2	16	0.3010	1.2041	0.0906	0.3624
3	54	0.4771	1.7323	0.2276	0.8261
4	128	0.6020	2.1072	0.3624	1.268
5	250	0.6989	2.3979	0.4884	1.67
		2.079	7.7425	1.169	4.

$\Sigma Y = nA + b\Sigma X \longrightarrow (1)$

$\Sigma XY = A\Sigma X + b\Sigma X^2 \longrightarrow (2)$

$7.7425 = 5A + 2.079b \longrightarrow (3) \times 2.079$

$4.1331 = 2.079A + ~~13.2815~~ 1.169b \longrightarrow (4) \times 5$

Unit - II Correlation

Correlation:- In measures of central tedency and dispersion to deal with data on single variable. The measures of central tedency from dispersion provides us with the information regarding one variable only. But in many business problems we deal with two variables which may have the relationship b/w them.

For ex:- Statistical data on demand and prices, hight and weight, saving and investment etc.

Correlation is a correlation between the two or more variables.

Types of correlation:- The relationship between two random variables x and y is called correlation. They are two types of correlation

* positive correlation
* Negative correlation

Positive correlation:- Correlation is a set to be positive when the values of two variables move in a same direction. So, that an increased or decreased values of one variable is increased the other variable is increased.

Ex1:- ~~Hight~~ Height and weight of a group of persons.

Ex2:- Demand and supply of a commandity.

Negative correlation:- Correlation is a set to be negative if both the values of the variables move in opposite direction

Ex1:- Traffic facilities (↑) – No.of accidents (↓)

Ex2:- Health facilities (↓) – No.of deaths (↑)

Simple, multiple and partial correlation :-

Correlation is set to be simple when only the relationship between two variables are taken, if more than two variables are taken simentienously then the correlation is set to be multiple correlation.

ex: when we study the relation ship between price and demand and supply of an optical

The correlation is set to be partial when we study the correlation between two are more variables then the influence of some other variables.

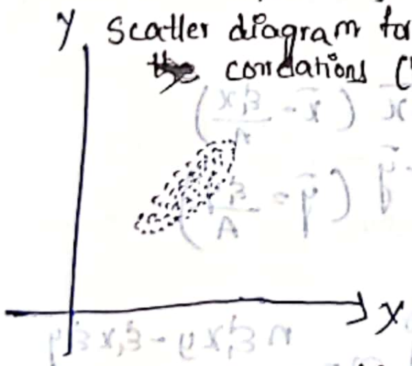
Methods of correlation (or) Methods of correlation co-efficient

The commonly used method for studying the correlation between two variables are :-

1. Scatter diagram method
2. karlpearson's correlation coefficient
3. Rank correlation coeffient

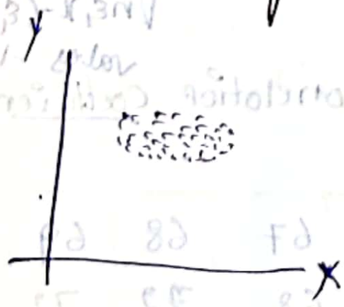
Scatter diagram Method :- The diagramatic presentation of a byvarient data $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$ is called scatter diagram. It is a simple graphical representation by plotting values of the random variables x on x axis and y on y axis.

Scatter diagram for +ve correlations $(r>0)$

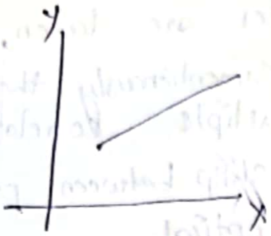


Scatter diagram for -ve correlation $(r<0)$

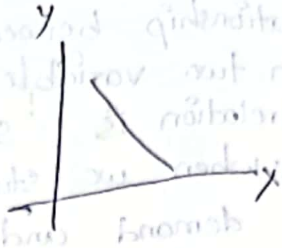
Scatter diagram for no correlation $(r=0)$



Scatter diagram perfect +ve correlation ($r=1$)



Scatter diagram perfect -ve correlation ($r=-1$)



Sol

karlpearson's correlation coffecient :-

As a measure of identity or degree of line relationship between two variable was suggested by a british a biometrician and statistician karlpearson. It is widely used method in practice. It is also known as "Pearson's correlation coffection"

The karlpearson's correlation coffecient is denoted by " r (or) r_{xy} (or) $r(x,y)$ "

IMPS

Formulas

$$r = \frac{cov(x,y)}{\sqrt{v(x)}, \sqrt{v(y)}} \quad (or) \quad r = \frac{cov(x,y)}{\sigma_x \, \sigma_y} \quad (or)$$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2} \cdot \sqrt{\Sigma y^2}}$$ where $x = x_i - \bar{x}$ ($\bar{x} = \frac{\Sigma x}{n}$)

$y = y_i - \bar{y}$ ($\bar{y} = \frac{\Sigma y}{A}$)

$$r = \frac{N\Sigma xy - \Sigma x \cdot \Sigma y}{\sqrt{N\Sigma x^2 - (\Sigma x)^2} \cdot \sqrt{N\Sigma y^2 - (\Sigma y)^2}} \quad (or) \quad r = \frac{n\Sigma xy - \Sigma x \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}, \sqrt{n\Sigma}}$$

values -1 to +1

Calculate (karlpearson's) correlation coefficient for the following data

Father	65	66	67	67	68	69	70	7
Son	67	68	65	68	72	72	69	

x	y	$x = x_i - \bar{x}$	$y = y_i - \bar{y}$	xy	x^2	y^2
65	67	-3	-2	6	9	4
66	68	-2	-1	2	4	1
67	65	-1	-4	4	1	16
67	68	-1	-1	1	1	1
68	72	0	3	0	0	9
69	72	1	3	3	1	9
70	69	2	0	0	4	0
72	71	4	2	8	16	4
$\Sigma x = 544$	$\Sigma y = 552$			$\Sigma xy = 24$	$\Sigma x^2 = 36$	$\Sigma y^2 = 44$

$n = 8$ $\quad \bar{x} = \frac{\Sigma x_i}{n}$, $\bar{y} = \frac{\Sigma y_i}{n}$

$\bar{x} = \frac{544}{8}$ $\quad \bar{y} = \frac{552}{8}$ $\quad 68 - 67 = -3 \rightarrow x = x_i = x - \text{value}$

$\bar{x} = 68$ $\quad \bar{y} = 69$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2} \cdot \sqrt{\Sigma y^2}}$$

$$r = \frac{24}{\sqrt{36} \cdot \sqrt{44}}$$

$$r = \frac{24}{6(6.6332)}$$

$$r = \frac{24}{39.7992}$$

$$\boxed{r = 0.6030}$$

(2) calculate Karlpearson's correlation coefficient for the between ages of husband & wife? $r = 0.9958$

Age of husband	23	27	28	29	30	31	33	35	36	39
Age of wife	18	22	23	24	25	26	28	29	30	32

x	y	$x_i - \bar{x}$	$y_i - \bar{y}$			
23	18	-8.1	-7.7	62.37	65.61	59.29
27	22	-4.1	-3.7	15.17	16.81	~~22.?~~ 13.69
28	23	-3.1	-2.7	8.37	~~10.95~~ 9.61	~~10.~~ 7.29
29	24	-2.1	-1.7	0.77	4.41	2.89
30	25	-1.1	0.7	0.77	1.21	0.49
31	26	0.1	0.3	0.03	0.01	0.09
33	28	1.9	2.3	4.37	3.61	5.29
35	29	3.9	3.3	12.87	15.21	10.89
36	30	4.9	4.3	21.07	24.01	18.49
39	32	7.9	6.3	~~49.77~~ 49.77	~~56.95~~ 62.41	39.69
$\Sigma x = 311$	$\Sigma y = 257$			$\Sigma xy = $ ~~176.93~~ 178.36	$\Sigma x^2 = 202.9$	$\Sigma y^2 = 158.1$

$$\bar{x} = \frac{\Sigma x_i}{n} \qquad \bar{y} = \frac{\Sigma y_i}{n}$$

$$\bar{x} = \frac{311}{10} \qquad \bar{y} = \frac{257}{10}$$

$$\bar{x} = 31.1 \qquad \bar{y} = 25.7$$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2} \cdot \sqrt{\Sigma y^2}} = \frac{178.36}{\sqrt{202.9} \cdot \sqrt{158.1}} = ~~0.9878~~$$

~~r = 0.9878~~

$r = 0.9958$

Calculate karpearson Correlation Coffecient betwe[en] ... from the following data [Take 69 as mean fo[r x] 112 for y

0.9355 0.7804

$\frac{\sum xy}{\sqrt{\sum x^2} \cdot \sqrt{\sum y^2}}$ = $\frac{185}{\sqrt{385} \cdot \sqrt{192}}$ = $\frac{185}{19.6214 \cdot 13.8564}$ = $\frac{185}{211.9609}$

= 0.6802

Correlation in bivariate frequency Table :-

In byvariate distribution, the data are fairly large, there vary summarized in the form of two way table. Here for each variable, the values are grouped into various keeping in view. The same consideration as in the case of univariate distribution.

Ex:- If there are "M" class for the x variable is and "n" classes for the "y" variable series, then there will be M n (class) cell in the following table.

why going different ways of values x and y using tally marks. we can find frequency for each cell and has thus oftene obtained so called bi-variate frequency table.

y \ x →	Class mid point $x_1, x_2 - - - - x - - - - x_m$	Total of y
Classes of mid values $y_1, y_2 - - - y - - - y_n$	f(x,y)	f_y
Total of x	f_x	N

The formula for computing the correlation co-efficient between x & y for bi-variate frequency table

$$r_{uv} = N \sum f_{uv}$$

Calculate the Karl pearson co-efficient correlation from the following data

x/y	18	19	20	21	22
20-25	3	2	-	-	-
15-20	-	5	4	-	-
10-15	-	-	7	10	-
5-10	-	-	-	3	2
0-5	-	-	-	3	1

$r_{uv} = 0.8373$ (-113)

↳ we have multiply the $3 \times U \times V$ variable given Question

v	u:	-2	-1	0	1	2				
	y↓ \ x→	18	19	20	21	22	f	fv	fv^2	fuv
-2	20-25	3 [12]	2 [4]	-	-	-	5	-10	20	16
-1	15-20		5 [15]	4 [10]	-	-	9	-9	9	5
0	10-15	-	-	7 [10]	10 [10]	-	17	0	0	0
1	5-10	-	-	-	3 [13]	2 [14]	5	5	5	7
2	0-5	-	-	-	3 [6]	1 [4]	8	8	16	10
	f	3	7	11	16	3	(40)	-6	50	38
	fu	-6	-7	0	16	6	9			
	fu^2	12	7	0	16	12	47			
	fuv	12	9	0	9	8	38			

$$r_{uv} = \frac{N\sum fuv - \sum fu \cdot \sum fv}{\sqrt{N\sum fu^2 - (\sum fu)^2} \cdot \sqrt{N\sum fv^2 - (\sum fv)^2}} = \frac{40(38) - 9 \times -6}{\sqrt{40 \times 47 - (9)^2} \cdot \sqrt{40 \times 50 - (-6)^2}} = \frac{1520 - (-54)}{\sqrt{1880 - 81} \cdot \sqrt{2000 - 36}}$$

$$= \frac{1574}{1879.\ldots} = 0.8373$$

Properties of karl pearson correlation co-efficients

Limits for karlpearson correlation co-efficient

Statement :- The karlpearson correlation co-efficient always lie in between -1 and +1

Proof :- We have

$$r = \frac{cov(xy)}{\sqrt{v(x)}\sqrt{v(y)}}$$

$$Cov(xy) = \frac{1}{n}\sum (xi-\bar{x})(yi-\bar{y})$$

$$v(x) = \frac{1}{n}\sum (xi-\bar{x})^2$$

$$v(y) = \frac{1}{n}\sum (yi-\bar{y})^2$$

$$Cov(xy) = \frac{1}{n}\sum aibi$$

$$v(x) = \frac{1}{n}\sum ai^2$$

$$v(y) = \frac{1}{n}\sum bi^2$$

$$r = \frac{\frac{1}{n}\sum aibi}{\sqrt{\frac{1}{n}\sum ai^2}\sqrt{\frac{1}{n}\sum bi^2}}$$

$$r = \frac{\frac{1}{n}\sum aibi}{\frac{1}{n}\sqrt{\sum ai^2}\sqrt{\sum bi^2}}$$

$$r = \frac{\sum aibi}{\sqrt{\sum ai^2}\sqrt{\sum bi^2}}$$

Squaring on both sides

$$r^2 = \frac{\sum ai^2 bi^2}{\sum ai^2 \sum bi^2}$$

According to cauchy - Sohwartz inequality

$$\sum (aibi)^2 \leq \sum ai^2 \sum bi^2 \Rightarrow \frac{\sum (aibi)^2}{\sum ai^2 \sum bi^2} \leq 1$$

$$\frac{\sum ai^2 bi^2}{\sum ai^2 \sum bi^2} \quad r^2 \leq 1 \quad r \leq +1 \quad -1 \leq r \leq +1$$

Statement :- karl pearson co-efficient of correlation is independent of origin & scale

proof :- let the origin & scale of variables x and y can be taken as

$$U = \frac{X-A}{h} \quad V = \frac{Y-B}{k} \; h, k > 0$$

In order to prove that karl pearson co-effecient is independent of origin and scale we have to prove that $r_{xy} = r_{uv}$

$$U = \frac{X-A}{h} \quad V = \frac{Y-B}{k}, k > 0$$

$$X = A + hu \qquad Y = B + kV$$

Taking expectations

$$E(x) = A + hE(u)$$

$$E(y) = B + kE(v)$$

$$x - E(x) = A + hu - A - hE(u)$$

$$= hu - hE(u)$$

$$x - E(x) = h[u - E(u)]$$

$$y - E(y) = B + kv - B - kE(v)$$

$$= kv - kE(v)$$

$$y - E(y) = k[v - E(v)]$$

Consider,

$$cov(xy) = E[(x - E(x))(y - E(y))]$$

$$cov(xy) = E[h(u - E(u))k(v - E(v))]$$

$$cov(xy) = hk\,E[(u - E(u))(v - E(v))] \text{ — from } ①$$

Consider,

$$V(x) = E(x - E(x))^2$$

$$V(x) = E[h(u - E(u))]^2$$

$$V[x] = h^2 E[u - E(u)]^2 \longrightarrow \text{from } ②$$

$$V(y) = h^2 E(v - E(v))^2 \longrightarrow ③$$

From ①, ② and ③

Rank Correlation coefficient Method:

Some times we come across Statistical Series in which the variables under consideration are not capable of quantitiave measurement. But, it can be arranged in Serial order this happens we we are dealing with qualitative characterstics such as honsety, beauty, character etc. This which can't be measured quantitatively but it can be arranged serially in such situations karlpearson correlation coefficient can't be used as a such cases Sphere man a british sychologist developed a formula in 1904. which consists in obtaining the correlation coefficient between the ranks of n individuals in the two attributes under studying.

Sphereman Rank correlation coefficient is usually denoted by 'Row' ρ it is given by and is given by

$$\rho = 1 - \frac{6 \sum d_i^2}{n(n^2-1)}$$ Non-repeated

In the case of Tied ranks we use the following formula.

$$\rho = 1 - \frac{6\left[\sum d_i^2 + \frac{m_1(m_1^2-1)}{12} + \frac{m_2(m_2^2-1)}{12} \dots\right]}{n(n^2-1)}$$

where $m_1, m_2 \dots$ is the no. of times a rank is re

① Find the rank correlation coefficient from the following

Marks in maths	123	125	130	110	105	134	121	106
Marks in stats	80	91	99	71	61	81	70	59

x	y	R_1	R_2	$d_i = R_1 - R_2$	d_i^2
123	80	4	4	0	0
125	91	3	2	1	1
130	99	2	1	1	1
110	71	6	5	1	1
105	61	8	7	1	1
134	81	1	3	-2	4
121	70	5	6	-1	1
106	59	7	8	-1	1
					$\Sigma d_i^2 = 10$

$\rho = 1 - \frac{6 \Sigma d_i^2}{n(n^2-1)}$

$\rho = 1 - \frac{6(10)}{8(8^2-1)}$

$\rho = 1 - \frac{60}{8(63)}$

$\rho = 1 - \frac{60}{504}$

$\rho = 1 - 0.1190$

$\boxed{\rho = 0.881}$

② Find 'ρ' from the following data

x	39	65	62	90	82	75	25	98	36	78
y	47	53	58	86	62	68	60	91	51	84

x	y	R_1	R_2	$d_i = R_1 - R_2$	d_i^2
39	47	8	10	-2	4
65	53	6	8	-2	4
62	58	7	7	0	0
90	86	2	2	0	0
82	62	3	5	-2	4
75	68	5	4	1	1
25	60	10	6	4	16
98	91	1	1	0	0
36	51	9	9	0	0
78	84	4	3	1	1
					$\Sigma d_i = 30$

$\rho = 1 - \frac{6 \Sigma d_i^2}{n(n^2-1)}$

$\rho = 1 - \frac{6(30}{10(10^2}$

$\rho = 1 - \frac{18}{10(}$

$\rho = 1 - -$

$\rho = 1 -$

$\boxed{\rho = 0.}$

③ 10 competetors in a beauty contest are ra
3 Judges as follows calculate ρ ?

karl pearson cc	Rank cc
1) It finds the linear relationship b/w 2 variables.	1) It finds the association b/w 2 variables
2) It is based on the assumption of the normality of the data	2) It is not based on the assumption of the normality of the data.
3) It can be applicable to bi-variable frequency distribution	3) It can not be applicable to bi-variable frequency distribution.
4) Small variation to the data includes similar variations in co-efficience	4) Any variation in the data which does not affect the co-efficient
5) It is coefficient lie is b/w -1 and +1	5) It's co-efficient also lie in b/w -1 and +1
6) It is most common method in use	6) It is used in some particular
7) It is denoted by 'r' and is given by $r = \frac{\sum xy}{\sqrt{\sum x^2}\sqrt{\sum y^2}}$	7) It is denoted by ρ and is given by $\rho = 1 - \frac{6\sum di^2}{n(n^2-1)}$

Derivation of rank correlation coefficient :-

Derive the Spearman's rank correlation coefficient

Proof :- Let $(x_1 y_1)(x_2 y_2) \ldots (x_n y_n)$ be the ranks of 'n individuals in the two characteristics A and B respectively.

Let $d_i = x_i - y_i$ denotes the diff b/w the ranks

we have $\bar{x} = \bar{y} = \frac{1+2+3+\ldots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$

$\sigma_x^2 = \sigma_y^2 = \frac{1}{n}(1^2+2^2+3^2+\ldots+n^2) - (\bar{x})^2$

$= \frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{n+1}{2}\right)^2 \Rightarrow \frac{(n+1)(2n+1)}{6}$

$$\frac{n+1}{2}\left(\frac{2n+1}{3}-\frac{n+1}{2}\right)$$

$$=\frac{n+1}{2}\left(\frac{4n+2-3n-3}{6}\right)$$

$$=\frac{n+1}{2}\left(\frac{n-1}{6}\right)$$

$$\sigma_x^2=\sigma_y^2=\frac{n^2-1}{12}$$

$(a+b)(a-b)=a^2-1$

$(n+1)(n-1)=n^2-1$

$=n^2-1$

we have $d_i = x_i - y_i$

Add and Sub $\bar{x}$

$$d_i = x_i-\bar{x}+\bar{x}-y_i$$

$$=(x_i-\bar{x})-(y_i-\bar{y})$$

$$d_i^2=[(x_i-\bar{x})-(y_i-\bar{y})]^2$$

$$=[(x_i-\bar{x})^2+(y_i-\bar{y})^2-2(x_i-\bar{x})(y_i-\bar{y})]$$

Taking Σ and $\div$ by on bs

$$\frac{\Sigma d_i^2}{n}=\frac{\Sigma[(x_i-\bar{x})^2+(y_i-\bar{y})^2-2\Sigma(x_i-\bar{x})(y_i-\bar{y})]}{n}$$

$$\frac{\Sigma d_i^2}{n}=\frac{\Sigma(x_i-\bar{x})^2}{n}+\frac{\Sigma(y_i-\bar{y})^2}{n}-\frac{2\Sigma(x_i-\bar{x})(y_i-\bar{y})}{n}$$

$$\frac{\Sigma d_i^2}{n}=\frac{\Sigma(x_i-\bar{x})^2}{n}+\frac{\Sigma(y_i-\bar{y})^2}{n}-\frac{2\Sigma(x_i-\bar{x})(y_i-\bar{y})}{n}$$

$$\frac{\Sigma d_i^2}{n}=\sigma_x^2+\sigma_y^2-2\,\mathrm{cov}(x,y)$$

$$=\sigma_x^2+\sigma_y^2-2r\sigma_x\sigma_y$$

$$r=\frac{\mathrm{cov}(x,y)}{\sigma_x\sigma_y}$$

$$=\sigma_x^2+\sigma_x^2-2r\sigma_x\sigma_y$$

$$r\sigma_x\sigma_y=\mathrm{cov}(x,y)$$

$$=2\sigma_x^2-2r\sigma_x^2$$

$$\frac{\Sigma d_i^2}{n}=2\sigma_x^2(1-r)$$

$$\Rightarrow 1-r=\frac{\Sigma d_i^2}{n\,2\sigma_x^2}$$

$$\Rightarrow 1-r=\frac{\Sigma d_i^2}{2n\left(\frac{n^2-1}{12}\right)}$$

$$r=1-\frac{6\Sigma d_i^2}{n(n^2-1)}$$

$$1-r=\frac{6\Sigma d_i^2}{n(n^2-1)}$$

$$\rho=1-\frac{6\Sigma d_i^2}{n(n^2-1)}$$

3. Additional concepts of ...

Co-efficient of concurrent deviation :-

The concurrent deviation method of correlation is based on the direction of changing in two variables. The co-efficient of concurrent deviation is denoted by r_c (or) r_c, and given by

$$r_c = \pm\sqrt{\pm\left(\frac{2c-n}{n}\right)}$$

where c = no. of +ve signs after multiplying the change of direction of "x"- series and y- series called concurrent deviations.

n = no. of pairs of observation is computed

Procedure of calculating co-efficient of concurrent deviation

→ Find out the change of direction in the present value as compared to previous one its x variable. If second value is more than the first put a positive (+) sign, if it is less than put a negative (-) sign and if it is equal put a zero "0" (or) equal = (0 (or) =). Repeat the same procedure for values in x series and denoted it by "d_x". If similarly find out the direction or change in y- series and denoted it by "dy".

→ Multiply d_x and corresponding dy and determine number of positive signs called concurrent deviation and denote it by "c".

→ The significant of ± signs outside and inside square root is, if the value of $\left(\frac{2c-n}{n}\right)$ is negative then keep negative as well as in square root so as to make positive. If the value of $\left(\frac{2c-n}{n}\right)$ is positive, then we ...

calculate r_c to the following data

x	150	154	160	172	160	165	180
y	200	180	170	160	190	180	172

calculate r_c to the following data

calls	320	310	300	290	282	270	240	210	245	360	360 / 330
Reverse	24	23	20	17	15	13	10	10	16	24	27 / 23

2)

X	Y	dx	dy	dx dy
20	25			
15	20	-	-	+
15	20	=	=	+
10	9	-	-	+
12	8	+	-	-
30	18	+	+	+
40	23	+	+	+
45	32	+	+	+
50	43	+	+	+
60	49	+	+	+
64	55	+	+	+
67	52	+	-	

$n = 11, \ c = 9$

$r_c = \pm\sqrt{\pm\frac{(2c-n)}{n}}$

$r_c = \pm\sqrt{\pm\frac{(2(9)-11)}{11}}$

$r_c = \pm\sqrt{\pm\frac{7}{11}}$

$r_c = \sqrt{0.6363}$

$r_c = 0.7976$

3)

X	Y	dx	dy	dxdy
150	200			
154	180	+	-	-
160	170	+	-	-
172	160	+	-	-
160	190	-	+	-
165	180	+	-	-
180	172	+	-	-

$r_c = \pm\sqrt{\pm\frac{(2c-n)}{n}}$

$r_c = \pm\sqrt{\pm\frac{(2(0)-6)}{6}}$

$r_c = \frac{-6}{6} = 1$

X	Y	dx	dy	dxdy
320	24			+
310	23	-	-	+
300	20	-	-	+
290	17	-	-	+
282	15	-	-	+
270	13	-	-	+
240	10	-	-	+
210	10	-	=	-
245	16	+	+	+
300	24	+	+	+
360	27	+	+	+
330	23	-	-	+

$C = 10$
$n = 11$

$r_c = \pm\sqrt{\pm\left(\frac{2C-n}{n}\right)}$

$r_c = \pm\sqrt{\pm\frac{2(10)-11}{11}}$

$r_c = \frac{9}{11}$

$r_c = \sqrt{0.8181}$

$r_c = 0.9044$

3. The pr... reliability probable...
4. within ... is ve... co-effic... In ... of ...

Standard error of ~~Coorlation~~ correlation coefficient :- If 'r' is the correlation coefficient of a sample of 'n' pairs of observations, than standard error (S E) of correlation co-efficient are is given by $S.E(r) = \frac{1-r^2}{\sqrt{n}}$

Propable Error of correlation coefficient :- The propable error (P E) of Correlation coefficient is obtained by formula $P.E(r) = 0.6745 \cdot S.E(r)$

$P.E(r) = 0.6745 \cdot \left(\frac{1-r^2}{\sqrt{n}}\right)$

where "n" is size of the sample (relative to of observations).

Properties of probable error of correlation coeffi...

1. If $r < P.E(r)$, then correlation is not significant.

2. If $r > 6' P.E(r)$ then correlation is sig...

The probable error is used for testing the reliability of an observed correlation coefficient. probable error is useful to determine the limits limits within which the population correlation coefficient is very near to sample values.

Co-efficient of determination:-

In the regre regression analysis, the residual variance of y is

$$S_y^2 = \sigma_y^2(1-r^2)$$

the require residual variance of x is

$$S_x^2 = \sigma_x^2(1-r^2)$$

In this analysis if are $r = \pm 1$, then $S_x = S_y = 0$ an two regression lines coinside.

If $r = 0$, then $S_x^2 = \sigma_x^2$, $S_y^2 = \sigma_y^2$ in this regression analysis the quantity r^2 is called the "co-efficient of determination"

Intra-class correlation:- In the usually correlation two variable measure different characterstic.

For ex:- Price and demand of a commondity in which price is one variable and demand is another variable, but many practical situations particullarly in medical, bylogical and agricultural field, one may be intrested to know the correla: among the units of a group or family.

Ex: In an agricultural experiments the study m be foced on the correlation among the yield of the plots in a block if the samer ferti is applied. In the study of correlation whe the both the variables measure the same

Characterstics such correlation is

Correlation:

let us consider n groups of families $F_1, F_2, \ldots F_n$ with $k_1, k_2 \ldots k_n$ members respectively [Let x_{ij} ($i = 1, 2 \ldots n$, $j = 1, 2 \ldots n$) be the value of j^{th} members of the i^{th} family.

The pairs of observations of i^{th} family are $(x_{ij}, x_{i\ldots}$

The value of x_{ij} of n groups can be represented bel[ow] using the following the table.

x_{11}	x_{21} ----	x_{i1} ----	x_{n1}
x_{12}	x_{22} ----	x_{i2} ----	x_{n2}
⋮			
x_{1j}	x_{2j} ----	x_{ij} ----	x_{nj}
⋮			
$x_1 k_1$	$x_2 k_2$ ----	$x_i k_j$ ----	$x_n k_n$

The table representing n pairs of observations is called Intra-class correlation table and correlation called Intra-class correlation

Partiall and

Imp: By the defination.

$$R^2_{1.23} = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{23}r_{13}}{(1-r_{23})^2}$$

$$R^2_{2.13} = \frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{(1-r_{13}^2)}$$

$$R^2_{3.12} = \frac{r_{13}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{(1-r_{12}^2)}$$

4m Imp

Properties of multiple correlation coefficient

① $0 \le R_{1.23} \le 1$

② If $R_{1.23} = 1$

③ If $R_{1.23} = 0$ – Then total partial correlation coefficient x_1 & 0

④ If $R_{1.23} \ge r_{12}\ r_{13}\ r_{23}$

② Then the association is perfect

① Calculate $R_{1.23}$, $R_{2.13}$, $R_{3.12}$ for the following.

$r_{12} = 0.6$, $r_{13} = 0.7$, $r_{23} = 0.65$

Sol: Given $r_{12} = 0.6$, $r_{13} = 0.7$, $r_{23} = 0.65$

$$R^2_{1.23} = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{23}r_{13}}{1 - r_{23}^2}$$

$$R_{1.23} = \frac{(0.6)^2 + (0.7)^2 - 2(0.6)(0.65)(0.7)}{(1 - 0.65^2)}$$

$$R_{1.23} = \frac{0.36 + 0.49 - 2(0.273)}{1 - 0.4225}$$

$$R_{1.23} = \frac{0.85 - 0.546}{0.5775} = \frac{0.304}{0.57}$$

Chapter - 4 Partial and Multiple

partial correlation :-

Correlation between any two variables study partially is studied after eliminating the linear effect of others then it is called as partial correlation.

In fact it remains the partial relationship only between two variables when the effect of other variable is excluded. Here we shall consider the partial correlation b/w two variables by eliminating the linear effect of third one.

Co-efficient of partial correlation :-

The correlation coefficient b/w two variables after the linear effect of third on each of them has been eliminated is called "partial correlation co-efficient".

The partial correlation coefficient between two variables usually denoted by $r_{12.3}$, $r_{13.2}$ and $r_{23.1}$ are given by

$$r_{12.3} = \frac{r_{12} - r_{13}\, r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

$$r_{23.1} = \frac{r_{23} - r_{21}\, r_{31}}{\sqrt{(1-r_{21}^2)(1-r_{31}^2)}}$$

$$r_{13.2} = \frac{r_{13} - r_{12}\, r_{32}}{\sqrt{(1-r_{12}^2)(1-r_{32}^2)}}$$

Where r_{12}, r_{13}, r_{23} are the correlation co-efficient X_1 and X_2 ; X_1 and X_3 ; X_2 and X_3 respectively.

$$r_{13.2} = \frac{r_{13} - r_{12}\, r_{32}}{\sqrt{(1-r_{12}^2)(1-r_{32}^2)}}$$

$$r_{13.2} = \frac{0.61 - 0.71(0.4)}{\sqrt{(1-0.71^2)(1-0.4^2)}}$$

$$r_{13.2} = \frac{0.61 - 0.284}{\sqrt{(1-0.5041)(1-0.16)}}$$

$$r_{13.2} = \frac{0.326}{\sqrt{0.4959(0.84)}}$$

$$r_{13.2} = \frac{0.326}{\sqrt{0.4165}} = \frac{0.326}{0.6453}$$

$$\boxed{r_{13.2} = 0.5051}$$

Multiple correlation :- whenever we are intrested to studying the joint effect of a group of variables not included in the group then it is called as "multiple correlation".

For ex :- yield of a crop say x_1 depends upon quality of seed x_2, fertility of soil x_3, fertilizers used x_4, irrigation facilities x_5, whether conditions x_6, and so on. To study the joint effect of fertilizers and irrigation facilities on yield of crop, we use multiple correlation.

Co-efficient of multiple correlation :- In a trivariate distribution in which each of the variable x_1, x_2 ... (so on) has one observation, the multiple correlation coefficient of x_1 on x_2 and x_3 usually denoted by $R_{1.23}$ i.e the simple correlation efficient b/w x_1 and joint effect of x_2 and x_3 on x_1. In words $R_{1.23}$ is the correlation coefficient b/w x_1 has given by the place x_1 on x_2 and x_3.

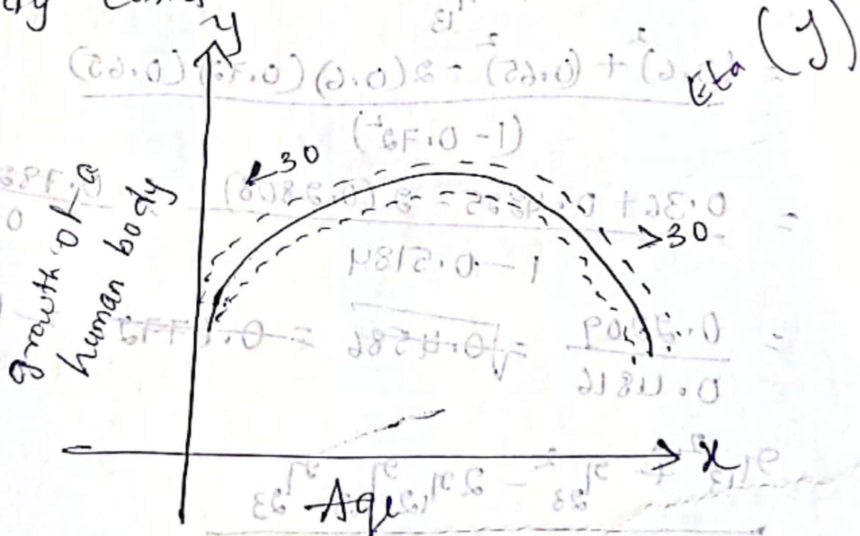
it's estimated value of regression of

Coorelation Ratio :-

Definition of C.R :- If the variables are linearly related K.p.c.c is the best measure to know the association b/w them. If the variables are not linearly related that is(i.e) if there is a curve line relationship b/w them, then the C.C is not an appropriate measure.

For ex:- In a group of persons we may find that there are many persons of the same weight but diff in height.

Correalation ratio can be explained through this ex. if we observe the growth of human body with incr. ase in the age of a group of persons in a locality. Generally up to a certain age (30) the parts of the human body grow up. After this age growth gradually come down at the old age, growth of the body comes to low level.



C.R can be carried out by two methods :-

1. Correlation Ratio of x on y

$$\eta_{xy} = \frac{\sum_{j=1}^{n} \left(\frac{T_j^2}{n_j}\right) - \frac{(\sum T_j)^2}{N}}{\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij} u_j^2 - \frac{(\sum T_j)^2}{N}}$$

2. Correlation Ratio of y on x.

UNIT - IV Regression Analysis

Introduction :- The literal meaning of regression means relationship b/w "average values." The regression concept was first introduced by Sir Francis Galton in 19^{th} century. Regression analysis means the estimation or prediction of the unknown value of one variable from the known value of another variable. It is one of the very important statistical tool which is extensively used in Physical social and economic etc. It is specially used in business and economics to study the relationship b/w two or more variables.

In Regression Analysis there are two types of Analysis which is to be predected called depedend variable or regressed variable and the variable which influence the values is used for predection called independented variable or regressor variable.

Simple linear Regression :- If two random variables and y are linearly related and mathematical function relationship is straight line, then this linear relationship b/w the random variables is called "Simple linear Regression."

Lines of Regression or Equations of Regression :- relationship can be measured with the help two regression lines they are

1) Reg Eqn. of x on y

2) Regression equation of y on x

Regression Equation of x on y :- The Regression eqⁿ of x on y is defined as $x-\bar{x} = r\frac{\sigma x}{\sigma y}(y-\bar{y})$

$$x-\bar{x} = b_{xy}(y-\bar{y})$$

where $b_{xy} = \frac{\Sigma xy}{\Sigma y^2}$

Regression eqⁿ of y on x :- The regression eqⁿ of y on x is defined as $y-\bar{y} = r\frac{\sigma y}{\sigma x}(x-\bar{x})$

(or)

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

Where $b_{yx} = \frac{\Sigma xy}{\Sigma x^2}$

Problem:

From the following data obtained two regression eqⁿ.

sales	91	97	108	121	67	124	51	73	111	57
purchase	71	75	69	97	70	91	39	61	80	47

x	y	$X = x-\bar{x}$	$Y = y-\bar{y}$	XY	X^2	Y^2
91	71	1	1	1	1	1
97	75	7	5	35	49	25
108	69	18	-1	-18	324	1
121	97	31	27	837	961	729
67	70	-23	0	0	529	0
124	91	34	21	714	1156	441
51	39	-39	-31	1209	1521	961
73	61	-17	-9	153	289	81
111	80	21	10	210	441	100
57	47	-33	-23	759	1089	529
$\Sigma x=900$	$\Sigma y=700$			~~80~~ 3900	6360	2868

$\bar{x} = \frac{\Sigma x}{n} = \frac{900}{10} = 90$; $\bar{y} = \frac{\Sigma y}{n} = \frac{700}{10} = 70$

Regression eqⁿ of x on y

$$x-\bar{x} = b_{xy}(y-\bar{y})$$

It identifies the nature of the variable i.e, which is independent and which is dependent variable. Regression co-efficient are not symmetric. $b_{xy} \neq b_{yx}$

5) correlation coefficient is a relative measure of the linear relationship and independent of the units of measurements. It is pure no b/w -1 and +1. i.e, correlation coefficient lines in b/w -1 to +1

5) The regression coefficient b_{xy} and b_{yx} or absolute measures. It measures change in value of one variable to changing the other variable.

Non-linear Regression

In practical situations regression may not be linear. In the study of regression, if the relationship b/w the two variables x and y is not linearly related, then the relation is curvilinear, the regression study for curvilinear relationship between the two or more variable is called "non-linear Regression". The non-linear regression can be formulated by $y = f(x, \beta) + \epsilon$.

Where y is the dependent are or predicited or estimated variable.

x is independent or predictor variable

B is parameter

E is Error term

There are several mathematical functional relationships for the fitting of non-linear or linear regression. The following are sum non-linear regression models.

1. second degree parabola

1. ax^2+bx+c

2. $a+bx+cx^2$

2. power curve $y = ax^b$

3. Exponetial curve 1. $y = ae^{bx}$

2. $y = ab^x$

Explain and No unexplain variation:-

In the regression analysis the total variation of the dependent values y are given by

$$\sum (y_i-\bar{y})^2$$

let $\hat{y}_i$ be the estimated value of y_i

consider

$$\sum (y_i-\bar{y})^2 = \sum (y_i-\hat{y}_i+\hat{y}_i-\bar{y})^2$$

$$\sum (y_i-\bar{y})^2 = \sum (y_i-\hat{y}_i)^2 + \sum (\hat{y}_i-\bar{y})^2$$

In ~~this~~ the expression ~~summation~~ $\sum (\hat{y}_i-\bar{y})^2$ explained variation the explained variation is sum of the squares of deviation b/w dependent value of y and $\hat{y}$.

The expression $\sum (y_i-\hat{y}_i)^2$ is un explained variation. the unexplained variation is the sum squares of the deviation from value of dependent variable y two its predicted or estimated value of y ($\hat{y}$).

Unit - V

Attributes

Theory of Attributes :-

Attributes means Quality or characterstics. The theory of attributes deals with the qualitative characteristic.

for ex:- The attributes are honesty, beauty, punctuallity, blindness, smooking, drinking etc.

An attributes may be mark by its presence or absence in the no. of given population. The presence or absence of an attribute may be regarded as change in value of a variable which can passes only two values 0 and 1.

suppose, the population is divided into two classes according to the presence or absence of single attribute. The presence of attribute is generally denoted is capital letters such as A, B, C, D ---- soon and the absence of attribute is denoted by small letters such as a, b, c, d ---- (or) α, β, γ ---- son.

Ex: if α represents the attribute sickness β represent bilndness and A and B represents healthy and non-bilndness respectively. The two classes the i.e is position of attributes and disposition of attributes are called "Complimentary Classes"

Dichotomous and Manifold Classification :- If the population is divided into two sub-classes Complementary Classes. the classification is said be dichotomous classification. A classification termed as manifold classification. if each class is further subdivided.

Class and class frequencies :-

Different attributes in themselves are called classes. The no. of observations (class) assigned to them are also called class frequencies. which are denoted by "( )."

Order of class and class frequencies :-

A class representing n attributes is called as n^{th} order.

i.e.

1. (A) is a class frequency of order 1
2. (AB), (αB), (BC) ---- are class frequency of order
3. (ABC) (αBγ), (AβC) --- are class frequency of order
4. and so on.

2x2 Contingency Table :-

	B	β	Total
A	(AB)	(Aβ)	(A)
α	(αB)	(αβ)	(α)
Total	(B)	(β)	N

formulas

$(A) = (AB) + (A\beta)$

$(\alpha) = (\alpha B) + (\alpha\beta)$

$(B) = (AB) + (\alpha B)$

$(\beta) = (A\beta) + (\alpha\beta)$

$N = (A) + (\alpha)$

(or)

$N = (B) + (\beta)$

(or)

$N = (AB) + (A\beta) + (\alpha B) +$

Problem

Given the following ultimate class frequencies the remaining frequency.

$(AB) = 471$, $(A\beta) = 151$, $(\alpha B) = 148$, $(\alpha\beta) = 230$

Sol: Given that

$(AB) = 471$, $(A\beta) = 151$, $(\alpha B) = 148$, $(\alpha\beta) = 230$

	B	β	Total
A	(AB) 471	(Aβ) 151	(A) = 622
α	(αB) 148	(αβ) 230	(α) = 378
Total	(B) 619	(β) 381	N = 1000

$(A) = (AB) + (A\beta) = 471 + 151 \Rightarrow 622$

$(\alpha) = (\alpha B) + (\alpha\beta) = 148 + 230 \Rightarrow 378$

$(B) = (AB) + (\alpha B)$

$= 471 + 148 = 619$

$(\beta) = (A\beta) + (\alpha\beta)$

$= 151 + 230 = 381$

$N = (A) + (\alpha)$

$= 622 + 378$

$N = 1000$

From the following set of positive frequencies obtained all other frequencies

$N = 64$, $(AB) = 9$, $(A) = 23$, $(B) = 13$

Sol: Given that

$N = 64$, $(AB) = 9$, $(A) = 23$, $(B) = 13$

	B	β	Total
A	9		23
α	13		N = 64
Total			

	B	β	Total
A	(AB) 9	(Aβ) 14	(A) 23
α	(αB) 4	(αβ) 37	(α) 41
Total	(B) 13	(β) 51	N 64

$(A) = (AB) + (A\beta)$

$(A\beta) = (A) - (AB)$

$= 23 - 9$

$= 14$

$(B) = (AB) + (\alpha B)$

$(\alpha B) = (B) - (AB) = 13 - 9 = 4$

$N = (A) + (\alpha)$

$(\alpha) = N - (A)$

$= 64 - 23$

$= 41$

$(\alpha) = (\alpha B) + (\alpha\beta)$

$(\alpha\beta) = (\alpha) - (\alpha B)$

$= 41 - 4$

$= 37$

$(\beta) = (A\beta) + (\alpha\beta)$

$= 14 + 37$

$= 51$

3X3 Contingency Table :-

	A		α		
	B	β	B	β	Total
C	(ABC)	(AβC)	(αBC)	(αβC)	(C)
γ	(ABγ)	(Aβγ)	(αBγ)	(αβγ)	(γ)
Total	(AB)	(Aβ)	(αB)	(αβ)	N

Formulas

$(C) = (ABC) + (A\beta C) + (\alpha BC) + (\alpha\beta C)$

$(\gamma) = (AB\gamma) + (A\beta\gamma) + (\alpha B\gamma) + (\alpha\beta\gamma)$

$(AB) = (ABC) + (AB\gamma)$

$(A\beta) = (A\beta C) + (A\beta\gamma)$

$(\alpha B) = (\alpha BC) + (\alpha B\gamma)$

$(\alpha\beta) = (\alpha\beta C) + (\alpha\beta\gamma)$

$N = (C) + (\gamma)$ (or) $N = (AB) + (A\beta) + (\alpha B) + ($

(or)

$N = (ABC) + (A\beta C) + (\alpha\beta C) + (\alpha BC) + (AB\gamma) + (A\beta\gamma) + (\alpha B\gamma) + (\alpha\beta\gamma)$

1) Given the following class frequencies find the frequencies.

$(ABC) = 149$, $(AB\gamma) = 738$, $(A\beta C) = 225$, $(A\beta\gamma) = 1196$, $(\alpha BC) = 204$

$(\alpha B\gamma) = 1762$, $(\alpha\beta C) = 171$, $(\alpha\beta\gamma) = 21482$

Sol:- Given that

$(ABC) = 149$, $(AB\gamma) = 738$, $(A\beta C) = 225$, $(A\beta\gamma) = 1196$, (αBC)

$(\alpha B\gamma) = 1762$, $(\alpha BC) = 171$, $(\alpha\beta\gamma) = 21482$

	A		α		
	B	β	B	β	Total
C	(ABC) 149	(AβC) 225	(αBC) 204	(αβC) 171	(C) 749
γ	(ABγ) 738	(Aβγ) 1196	(αBγ) 1762	(αβγ) 21482	(γ) 25178
Total	(AB) 887	(Aβ) 1421	(αB) 1966	(αβ) 21653	N 25927

$(C) = (ABC) + (A\beta C) + (\alpha BC) + (\alpha\beta C)$

$= 149 + 225 + 204 + 171$

$= 749$

$(\gamma) = (AB\gamma) + (A\beta\gamma) + (\alpha B\gamma) + (\alpha\beta\gamma)$

$(\gamma) = 738 + 1196 + 1762 + 21482$

$= 25178$

$(B) = (ABC) + (AB\gamma)$

$= 149 + 1196$ 738

$= 887$

$(A\beta) = (A\beta C) + (A\beta\gamma)$

$= 225 + 1196 \Rightarrow 1421$

$(\alpha B) = (\alpha BC) + (\alpha B\gamma)$

$= 204 + 1762$

$= 1966$

$(\alpha\beta) = (\alpha\beta C) + (\alpha\beta\gamma)$

$= 171 + 21482$

$= 21653$

$N = (C) + (\gamma)$

$N = 749 + 25178$

$N = 25927$

$\alpha\beta\gamma = (\gamma) - ((AB\gamma) + (A\beta\gamma) + (\alpha B\gamma))$

$= 22943 - (431 + 765 + 1249)$

$= 22943 - 2445$

$= 20498$

$(\gamma) = (N) - (C)$

$= 23713 - 770$

$= 22943$

$(\alpha\beta) = (\alpha\beta C) + (\alpha\beta\gamma)$

$= 169 + 20498$

$= 20667$

Find the following frequencies obtained all other class frequencies

(N) : 23713 (A) = 1618 , (B) = 2015 (C) = 770

(AB) = 587 (AC) = 422 (BC) = 335 (ABC) = 156

	A		α		
	B	β	B	β	Total
C	(ABC) 156	(AβC) 266	(αBC) 179	(αβC) 169	(C) 770
γ	(ABγ) 431	(Aβγ) 765	(αBγ) 1249 ~~7299~~	(αβγ)	(γ) 22943
Total	(AB) 587	(Aβ)	(αB)	(αβ)	(N) 23713

$(A\beta C) = NAC\beta = NAC[1-B]$

$= NAC - NABC$

$= (AC) - (ABC)$

$= 422 - 156$

$= 266$

$(\alpha BC) = NBC\alpha$

$= NBC(1-A)$

$= (BC) - (ABC)$

$= 335 - 156$

$= 179$

$(\alpha\beta C) =$ ~~$NB\alpha\gamma$~~ $NC\alpha\beta$

$=$ ~~$NBC[1-$~~ $NC[(1-A)(1-B)]$

$= NC[1 - B - A + AB]$

$= NC - NBC - NAC + NABC$

$= (C) - (BC) - (AC) + (ABC)$
$= 770 - 335 - 422 + 156$
$= 169$

$(AB\gamma) = NAB\gamma = NA(1-B)(1-C)$
$= NA[1-C-B+BC]$
$= (A) - (AC) - (AB) + (ABC)$
$= 1618 - 422 - 587 + 156$
$= 765$

$(\alpha B\gamma) = NB\alpha\gamma$
$= NB[(1-A)(1-C)]$
$= NB(1-C-A+AC)$
$= NB - NBC - NAB + NABC$
$= (B) - (BC) - (AB) + (ABC)$
$= 2015 - 335 - 587 + 156$
$= 1686 - 587 + 156$
$= 1093 + 156$
$= 1249$

Consistency of Data

In order to find out whether the data is consistent or not we have to apply a very simple test. The test is to findout whether one or more and altimate class frequencies is negative or not. If non of the class frequency negative then we can say the the given data is consistence otherwise the data is "inconsistent."

Conditions for consistency of data

The necessary and sufficient condition for the consistency of a set of independent class frequencies that is no altimate class frequency be positive.

I. Single attribute

For single attribute A we have conditions for consistency as follows

i) $(A) \geq 0$

ii) $(\alpha) \geq 0$ (or) $N \geq (A)$

II. Two attributes

For 2 attributes A and B the conditions are follow

i) $(AB) \geq 0$

ii) $(A\beta) \geq 0 \Rightarrow (A) \geq (AB)$

iii) $(\alpha B) \geq 0 \Rightarrow (B) \geq (AB)$

iv) $(\alpha\beta) \geq 0 \Rightarrow N \geq (A) + (B)$.

III. Three attributes

For three attributes A, B the conditions are follow

i) $(ABC) \geq 0$

ii) $(A\beta C) \geq 0 \Rightarrow (AC) \geq (ABC)$

iii) $(AB\gamma) \geq 0 \Rightarrow (AB) \geq (ABC)$

iv) $(A\beta\gamma) \geq 0 \Rightarrow (A) \geq (AB)$

v) $(\alpha BC) \geq 0 \Rightarrow (BC) \geq (ABC)$

vi) $(\alpha B\gamma) \geq 0 \Rightarrow (ABC) \geq (A$

vii) $(\alpha\beta C) \geq 0 \Rightarrow (ABC) \geq (AC)$

viii) $(\alpha\beta\gamma) \geq 0 \Rightarrow N \geq (A) + (B) + (AC) + (BC)$

Problems

1) Examine the consistency following data

$N = 150, (A) = 50, (B) = 40,$

Sol: Given that

N = 150, (A) = 50

(B) = 40, (AB) = 60

(i) (AB) ≥ 0 ⇒ 60 ≥ 0 true

(ii) (AB) ≥ 0 ⇒ (A) ≥ (AB)

⇒ 50 ≥ 60 False

(iii) (B) ≥ (AB) ⇒ 40 ≥ 60 False

(iv) N ≥ (A) + (B) − (AB)

⇒ 150 ≥ 50 + 40 − 60

⇒ 150 ≥ 30 true

The data is inconsistency.

Examing the consistency the following data:

N = 1000, (A) = 525, (B) = 312,

(C) = 470, (AB) = 412, (BC) = 86,

(AC) = 147, (ABC) = 25

Sol: Given that

N = 1000, (A) = 525, (B) = 312,

(C) = 470, (AB) = 412, (BC) = 86

(AC) = 147, (ABC) = 25

It is three attributes

(i) (ABC) ≥ 0

25 ≥ 0 ⟶ True

(ii) (AC) ≥ (ABC)

147 ≥ 412 False

(iii) (AB) ≥ (ABC)

412 ≥ 25 True

(iv) (ABC) ≥ (AB) + (AC) − (A)

25 ≥ 412 + 147 − 525

25 ≥ 559 − 525

25 ≥ 34

(v) (BC) ≥ (ABC)

86 ≥ 25 True

(vi) (ABC) ≥ (AB) + (BC)

25 ≥ 412 + 86

25 ≥ 498 False

(vii) (ABC) ≥ (AC) + (BC) − (C)

25 ≥ 147 + 86 − 470

25 ≥ 233 − 470

25 ≥ 237 False

(viii) N ≥ (A) + (B) + (C) − (AC) + (BC) + (ABC)

1000 ≥ 525 + 312 + 470 − 412 + 86 + 25

1000 ≥ 1307 − 523

1000 ≥ 784 True. It is Inconsistency

Homework

Given that

(N) = 23713, (A) = 1618, (B) = 2015

(C) = 770, (AB) = 587, (AC) = 422,

(BC) = 335, (ABC) = 156

(i) (ABC) ≥ 0

156 ≥ 0 ⟶ True

(ii) (AC) ≥ (ABC)

422 ≥ 156 ⟶ True

(iii) (AB) ≥ (ABC)

587 ≥ 156 ⟶ True

(iv) (ABC) ≥ (AB) + (AC) − (A)

156 ≥ 587 + 422 − 1618

156 ≥ 1009 − 1618

156 ≥ 609 ⟶ False

(v) (BC) ≥ (ABC)

335 ≥ 156 ⟶ True

vi) $(A\beta C) \geq (AB) + (\beta C)$

$156 \geq 587 + 335$

$156 \geq 922 \rightarrow$ False

vii) $(ABC) \geq (AC) + (BC) - (C)$

$156 \geq 422 + 335 - 770$

$156 \geq 757 - 770$

$156 \geq 13$ True

viii) $N \geq (A) + (B) + (C) - (AC) + (AB) + (ABC)$

$23713 \geq 1618 + 2015 + 770 - 422 + 587 + 335$

$23713 \geq 4403 - 1344$

$23713 \geq 3059 \rightarrow$ False

Association of attributes

In statistics two attributes are said to be associated if their occur together in a number of times. It is of three types they are

1) positive association
2) Negative association
3) Independent association

Positive association :-
The attributes are said to be in positive association when there are presence and absence together in the data.

Negative association :-
when the presence of one attribute is associated with the absence of other attribute then association is known as negative association.

Independent association
When the attribute do not have tendency to be present together do not cause presence of the other attribute the association is known as independent association.

The association can be measured by 4 methods

1. Frequency method
2. proportion method
3. yule's coefficient of association and colligation
4. contingency table

frequency Method
In this method we wi[ll] compare the observed f[requency] by (AB) or $(AB)_0$. And [the] expected frequency are [denoted] by $(AB)_e$

In frequency metho[d]

$$(AB)_e = \frac{(A)(B)}{N}$$

In the observed fr[equency] is greater than ex[pected] frequency that is (then the association [is] said to be as [positive] association.

(i) A in B & β

$\frac{(AB)}{(B)} = \frac{60}{135} = 0.4444$

$\frac{(A\beta)}{(\beta)} = \frac{13}{45} = 0.2888$

$\because \frac{(AB)}{(B)} > \frac{(A\beta)}{(\beta)}$

∴ Association is positive

(ii) α in B

$\frac{(\alpha B)}{(B)} = \frac{75}{135} = 0.5555$

$\frac{(\alpha\beta)}{(\beta)} = \frac{32}{45} = 0.7111$

$\frac{(\alpha B)}{(B)} < \frac{(\alpha\beta)}{(\beta)}$

∴ Association is negative

(iii) B in A & α :-

$\frac{(BA)}{(A)} = \frac{60}{73} = 0.8219$

$\frac{(B\alpha)}{(\alpha)} = \frac{75}{107} = 0.7009$

$\frac{(BA)}{(A)} > \frac{(B\alpha)}{(\alpha)}$

∴ Association is positive

(iv) β in A & α

$\frac{(\beta A)}{(A)} = \frac{13}{73} = 0.1780$

$\frac{(\beta\alpha)}{(\alpha)} = \frac{32}{107} = 0.2990$

$\therefore \frac{(\beta A)}{(A)} < \frac{(\beta\alpha)}{(\alpha)}$

∴ Association is negative

Yule's co-efficient of association & colligation

The frequency and proportion method tell us that kind of association but it does not use the nume measure of association. In this contest yule's ha made and to compute the association numeri Hence it is known as yule's coefficient of association and it is denoted as Q_{AB} & Q

If $Q_{AB} = 0$, A and B are independent if

If $Q_{AB} = +1$, A and B are completely associative.

$Q_{AB} = -1$ and A and B are disassociated

$$\therefore Q_{AB} \text{ or } Q = \frac{(AB)(\alpha\beta) - (\alpha B)(A\beta)}{(AB)(\alpha\beta) + (\alpha B)(A\beta)}$$

Yule's had developed the association and

$$Y = \frac{\sqrt{(AB)(\alpha\beta)} - \sqrt{(A\beta)(\alpha B)}}{\sqrt{(AB)(\alpha\beta)} + \sqrt{(A\beta)(\alpha B)}}$$

$$= \frac{\sqrt{7600} - \sqrt{16000}}{\sqrt{7600} - \sqrt{16000}}$$

$$= \frac{87.1779 - 126.4911}{87.1779 + 126.4911}$$

$$= \frac{-39.3132}{213.669}$$

$$= -0.1839.$$

Contigency Table :- An Attribute A is divided into "C" sub class and the an Attribute B is divided into "r" sub class. then the classification is known as manifold classification. The frequency of manifold classification is represented in a table, called contigency Table.

A/B ↓ →	A_1	A_2	A_3	...	A_i	...	A_c	Total
B_1	(A_1B_1)	(A_2B_1)	(A_3B_1)	---	(A_iB_1)	---	(A_cB_1)	(B_1)
B_2	(A_1B_2)	(A_2B_2)	(A_3B_3)	---	(A_iB_2)	-	(A_cB_2)	(B_2)
B_3	(A_1B_3)	(A_2B_3)	(A_3B_3)	---	(A_iB_3)	--	(A_cB_3)	(B_3)
⋮								
B_j	(A_1B_j)	(A_2B_j)	(A_3B_j)	---	(A_iB_j)	---	(A_cB_j)	(B_j)
⋮								
B_r	(A_1B_r)	(A_2B_r)	(A_3B_r)	---	(A_iB_r)	---	(A_cB_r)	(B_r)
Total	(A_1)	(A_2)	(A_3)	---	(A_i)	----	(A_2)	N

(A_iB_j) frequency of i^{th} subclass of A an j^{th} subclass of B. $(A_1), (A_2), (A_3)$ ---- (A_i) ---- (A_c) and $(B_1), (B_2)(B_3)$ -- (B_r) are marginal frequency of A_1, A_2, A_3

... $B_1, B_2, B_3 \ldots B_j \ldots B_r$ respectively. These contigency table we can reduce 2×2 contigency table. We can measure the association b/w attributes the following formulas are used.

1. χ^2 contigency :-

$$\chi^2 = \Sigma \left[\frac{(o_i - e_i)^2}{e_i} \right]$$

o_i = observed frequency ; e_i = expected frequency

2. Mean square contigency ϕ^2 :- Mean square contigency is denoted by ϕ^2. and it defined as $\phi^2 = \frac{\chi^2}{n}$.

3. pearson and Mean square contigency :- pearson & mean square contigency is denoted by "C". It is defined as $C = \sqrt{\frac{\phi^2}{1+\phi^2}}$

4. Tchprow's coefficient of contigency :- Tchprow's coefficient of contigency is denoted by "T." It is defined as $T = \sqrt{\frac{\phi^2}{(c-1)(r-1)}}$

where c = no. of columns ; r = no. of rows.

Calculate (i) χ^2 (ii) ϕ^2 (iii) C (iv) T

	very low	low	high	very high	Total
college	24 [43]	97 [74]	62 [57]	58 [65]	241
High school	22 [21]	28 [37]	30 [28]	41 [33]	121
middle school	32 [13]	10 [22]	11 [17]	20 [19]	73
Total	78	135	103	119	

$\frac{78 \times 241}{435}$

$$e_i = \frac{A_1 \times \text{Total}}{N}$$